5.1 Areas Between Curves

In Chapter 4 we defined and calculated areas of regions that lie under the graphs of functions. Here we use integrals to find areas of regions that lie between the graphs of two functions.

Consider the region \( S \) that lies between two curves \( y = f(x) \) and \( y = g(x) \) and between the vertical lines \( x = a \) and \( x = b \), where \( f \) and \( g \) are continuous functions and \( f(x) \geq g(x) \) for all \( x \) in \( [a, b] \). (See Figure 1.)

Just as we did for areas under curves in Section 4.1, we divide \( S \) into \( n \) strips of equal width and then we approximate the \( i \)th strip by a rectangle with base \( \Delta x \) and height \( f(x_i^*) - g(x_i^*) \). (See Figure 2. If we like, we could take all of the sample points to be right endpoints, in which case \( x_i^* = x_i \).) The Riemann sum

\[
\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x
\]

is therefore an approximation to what we intuitively think of as the area of \( S \).

This approximation appears to become better and better as \( n \to \infty \). Therefore we define the area \( A \) of the region \( S \) as the limiting value of the sum of the areas of these approximating rectangles.

\[
A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x
\]

We recognize the limit in (1) as the definite integral of \( f - g \). Therefore we have the following formula for area.

\[
A = \int_{a}^{b} [f(x) - g(x)] \, dx
\]

The area \( A \) of the region bounded by the curves \( y = f(x), y = g(x) \), and the lines \( x = a, x = b \), where \( f \) and \( g \) are continuous and \( f(x) \geq g(x) \) for all \( x \) in \( [a, b] \), is

\[
A = \int_{a}^{b} [f(x) - g(x)] \, dx
\]

Notice that in the special case where \( g(x) = 0 \), \( S \) is the region under the graph of \( f \) and our general definition of area (1) reduces to our previous definition (Definition 4.1.2).
In the case where both $f$ and $g$ are positive, you can see from Figure 3 why (2) is true:

$$A = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx = \int_a^b [f(x) - g(x)] \, dx$$

**EXAMPLE 1** Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

**SOLUTION** The region is shown in Figure 4. The upper boundary curve is $y = x^2 + 1$ and the lower boundary curve is $y = x$. So we use the area formula (2) with $f(x) = x^2 + 1$, $g(x) = x$, $a = 0$, and $b = 1$:

$$A = \int_0^1 (x^2 + 1) - x \, dx = \int_0^1 (x^2 - x + 1) \, dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x \bigg|_0^1 = \frac{1}{3} - \frac{1}{2} + 1 = \frac{5}{6}$$

In Figure 4 we drew a typical approximating rectangle with width $\Delta x$ as a reminder of the procedure by which the area is defined in (1). In general, when we set up an integral for an area, it’s helpful to sketch the region to identify the top curve $y_T$, the bottom curve $y_B$, and a typical approximating rectangle as in Figure 5. Then the area of a typical rectangle is $(y_T - y_B) \Delta x$ and the equation

$$A = \lim_{n \to \infty} \sum_{i=1}^n (y_T - y_B) \Delta x = \int_a^b (y_T - y_B) \, dx$$

summarizes the procedure of adding (in a limiting sense) the areas of all the typical rectangles.

Notice that in Figure 5 the left-hand boundary reduces to a point, whereas in Figure 3 the right-hand boundary reduces to a point. In the next example both of the side boundaries reduce to a point, so the first step is to find $a$ and $b$.

**EXAMPLE 2** Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

**SOLUTION** We first find the points of intersection of the parabolas by solving their equations simultaneously. This gives $x^2 = 2x - x^2$, or $2x^2 - 2x = 0$. Thus $2x(x - 1) = 0$, so $x = 0$ or $1$. The points of intersection are $(0, 0)$ and $(1, 1)$.

We see from Figure 6 that the top and bottom boundaries are

$$y_T = 2x - x^2 \quad \text{and} \quad y_B = x^2$$

The area of a typical rectangle is

$$(y_T - y_B) \Delta x = (2x - x^2 - x^2) \Delta x$$

and the region lies between $x = 0$ and $x = 1$. So the total area is

$$A = \int_0^1 (2x - 2x^2) \, dx = 2 \int_0^1 (x - x^2) \, dx$$

$$= 2 \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = 2 \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{1}{3}$$
Sometimes it’s difficult, or even impossible, to find the points of intersection of two curves exactly. As shown in the following example, we can use a graphing calculator or computer to find approximate values for the intersection points and then proceed as before.

**EXAMPLE 3** Find the approximate area of the region bounded by the curves
\[ y = \frac{x}{\sqrt{x^2 + 1}} \text{ and } y = x^4 - x. \]

**SOLUTION** If we were to try to find the exact intersection points, we would have to solve the equation
\[ \frac{x}{\sqrt{x^2 + 1}} = x^4 - x. \]

This looks like a very difficult equation to solve exactly (in fact, it’s impossible), so instead we use a graphing device to draw the graphs of the two curves in Figure 7. One intersection point is the origin. We zoom in toward the other point of intersection and find that \( x = 1.18 \). (If greater accuracy is required, we could use Newton’s method or solve numerically on our graphing device.) So an approximation to the area between the curves is
\[
A \approx \int_{0}^{1.18} \left( \frac{x}{\sqrt{x^2 + 1}} - (x^4 - x) \right) \, dx
\]

To integrate the first term we use the substitution \( u = x^2 + 1 \). Then \( du = 2x \, dx \), and when \( x = 1.18 \), we have \( u = 3.39 \); when \( x = 0 \), \( u = 1 \). So
\[
A \approx \frac{1}{2} \left[ \int_{1}^{3.39} \frac{u}{\sqrt{u}} \, du - \int_{0}^{1.18} (x^4 - x) \, dx \right]
\]
\[
= \sqrt{u} \bigg|_{1}^{3.39} - \left[ \frac{x^5}{5} - \frac{x^2}{2} \right]_{0}^{1.18}
\]
\[
= \sqrt{2.39} - 1 - \left( \frac{(1.18)^5}{5} + \frac{(1.18)^2}{2} \right)
\]
\[
\approx 0.785
\]

**EXAMPLE 4** Figure 8 shows velocity curves for two cars, A and B, that start side by side and move along the same road. What does the area between the curves represent? Use the Midpoint Rule to estimate it.

**SOLUTION** We know from Section 4.4 that the area under the velocity curve \( A \) represents the distance traveled by car A during the first 16 seconds. Similarly, the area under curve \( B \) is the distance traveled by car B during that time period. So the area between these curves, which is the difference of the areas under the curves, is the distance between the cars after 16 seconds. We read the velocities from the graph and convert them to feet per second (1 mi/h = \( \frac{5280}{3600} \) ft/s).

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_A )</td>
<td>0</td>
<td>34</td>
<td>54</td>
<td>67</td>
<td>76</td>
<td>84</td>
<td>89</td>
<td>92</td>
<td>95</td>
</tr>
<tr>
<td>( v_B )</td>
<td>0</td>
<td>21</td>
<td>34</td>
<td>44</td>
<td>51</td>
<td>56</td>
<td>60</td>
<td>63</td>
<td>65</td>
</tr>
<tr>
<td>( v_A - v_B )</td>
<td>0</td>
<td>13</td>
<td>20</td>
<td>23</td>
<td>25</td>
<td>28</td>
<td>29</td>
<td>29</td>
<td>30</td>
</tr>
</tbody>
</table>
We use the Midpoint Rule with \( n = 4 \) intervals, so that \( \Delta t = 4 \). The midpoints of the intervals are \( t_1 = 2, t_2 = 6, t_3 = 10, \) and \( t_4 = 14 \). We estimate the distance between the cars after 16 seconds as follows:

\[
\int_0^{16} (v_A - v_B) \, dt \approx \Delta t \left[ 13 + 23 + 28 + 29 \right] \\
= 4(93) = 372 \text{ ft}
\]

**EXAMPLE 5** Figure 9 is an example of a *pathogenesis curve* for a measles infection. It shows how the disease develops in an individual with no immunity after the measles virus spreads to the bloodstream from the respiratory tract.

The patient becomes infectious to others once the concentration of infected cells becomes great enough, and he or she remains infectious until the immune system manages to prevent further transmission. However, symptoms don’t develop until the “amount of infection” reaches a particular threshold. The amount of infection needed to develop symptoms depends on both the concentration of infected cells and time, and corresponds to the area under the pathogenesis curve until symptoms appear. (See Exercise 4.1.19.)

(a) The pathogenesis curve in Figure 9 has been modeled by \( f(t) = -t(t - 21)(t + 1) \). If infectiousness begins on day \( t_1 = 10 \) and ends on day \( t_2 = 18 \), what are the corresponding concentration levels of infected cells?

(b) The *level of infectiousness* for an infected person is the area between \( N = f(t) \) and the line through the points \( P_1(t_1, f(t_1)) \) and \( P_2(t_2, f(t_2)) \), measured in \((\text{cells/mL}) \cdot \text{days}\). (See Figure 10.) Compute the level of infectiousness for this particular patient.

**SOLUTION**

(a) Infectiousness begins when the concentration reaches \( f(10) = 1210 \text{ cells/mL} \) and ends when the concentration reduces to \( f(18) = 1026 \text{ cells/mL} \).
(b) The line through \( P_1 \) and \( P_2 \) has slope
\[
\frac{1026 - 1210}{18 - 10} = \frac{-184}{8} = -23
\]
and equation \( N - 1210 = -23(t - 10) \), or \( N = -23t + 1440 \). The area between \( f \) and this line is
\[
\int_{10}^{18} \left[ f(t) - (-23t + 1440) \right] dt = \int_{10}^{18} (-t^3 + 20t^2 + 21t + 23t - 1440) dt
\]
\[
= \int_{10}^{18} (-t^3 + 20t^2 + 44t - 1440) dt
\]
\[
= \left[ -\frac{t^4}{4} + 20\frac{t^3}{3} + 44\frac{t^2}{2} - 1440t \right]_{10}^{18}
\]
\[
= -6156 - (-8033\frac{1}{2}) = 1877
\]
Thus the level of infectiousness for this patient is about 1877 (cells/mL) · days.

If we are asked to find the area between the curves \( y = f(x) \) and \( y = g(x) \) where \( f(x) \geq g(x) \) for some values of \( x \) but \( g(x) \geq f(x) \) for other values of \( x \), then we split the given region \( S \) into several regions \( S_1, S_2, \ldots \) with areas \( A_1, A_2, \ldots \) as shown in Figure 11. We then define the area of the region \( S \) to be the sum of the areas of the smaller regions \( S_1, S_2, \ldots \), that is, \( A = A_1 + A_2 + \ldots \). Since
\[
|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \geq g(x) \\ g(x) - f(x) & \text{when } g(x) \geq f(x) \end{cases}
\]
we have the following expression for \( A \).

**Example 6** Find the area of the region bounded by the curves \( y = \sin x, y = \cos x, x = 0, \) and \( x = \pi/2 \).

**Solution** The points of intersection occur when \( \sin x = \cos x \), that is, when \( x = \pi/4 \) (since \( 0 \leq x \leq \pi/2 \)). The region is sketched in Figure 12.
Observe that $\cos x \geq \sin x$ when $0 \leq x \leq \pi/4$ but $\sin x \geq \cos x$ when $\pi/4 \leq x \leq \pi/2$. Therefore the required area is

$$
A = \int_0^{\pi/2} |\cos x - \sin x| \, dx = A_1 + A_2
$$

$$
= \int_0^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) \, dx
$$

$$
= \left[\sin x + \cos x\right]_0^{\pi/4} + \left[-\cos x - \sin x\right]_{\pi/4}^{\pi/2}
$$

$$
= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)
$$

$$
= 2\sqrt{2} - 2
$$

In this particular example we could have saved some work by noticing that the region is symmetric about $x = \pi/4$ and so

$$
A = 2A_1 = 2 \int_0^{\pi/4} (\cos x - \sin x) \, dx
$$

Some regions are best treated by regarding $x$ as a function of $y$. If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where $f$ and $g$ are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$ (see Figure 13), then its area is

$$
A = \int_c^d [f(y) - g(y)] \, dy
$$

If we write $x_R$ for the right boundary and $x_L$ for the left boundary, then, as Figure 14 illustrates, we have

$$
A = \int_c^d (x_R - x_L) \, dy
$$

Here a typical approximating rectangle has dimensions $x_R - x_L$ and $\Delta y$.

**EXAMPLE 7** Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

**SOLUTION** By solving the two equations we find that the points of intersection are $(-1, -2)$ and $(5, 4)$. We solve the equation of the parabola for $x$ and notice from Figure 15 that the left and right boundary curves are

$$
x_L = \frac{1}{2} y^2 - 3 \quad \text{and} \quad x_R = y + 1
$$
We must integrate between the appropriate y-values, $y = -2$ and $y = 4$. Thus

$$A = \int_{-2}^{4} (x_R - x_L) \, dy = \int_{-2}^{4} [(y + 1) - (\frac{1}{2}y^2 - 3)] \, dy$$

$$= \int_{-2}^{4} (-\frac{1}{2}y^2 + y + 4) \, dy$$

$$= \frac{1}{2} \left[ \frac{y^3}{3} \right]_{-2}^{4} + \frac{y^2}{2} \bigg|_{-2}^{4} + 4y_{-2}^{4}$$

$$= -\frac{1}{6} (64) + 8 + 16 - \left( \frac{16}{3} + 2 - 8 \right) = 18$$

\section*{NOTE} We could have found the area in Example 7 by integrating with respect to $x$ instead of $y$, but the calculation is much more involved. Because the bottom boundary consists of two different curves, it would have meant splitting the region in two and computing the areas labeled $A_1$ and $A_2$ in Figure 16. The method we used in Example 7 is much easier.

\section*{5.1 EXERCISES}

1–4 Find the area of the shaded region.

9. $y = \sqrt{x + 3}, \quad y = (x + 3)/2$

10. $y = \sin x, \quad y = 2x/\pi, \quad x \geq 0$

11. $x = 1 - y^2, \quad x = y^2 - 1$

12. $4x + y^2 = 12, \quad x = y$

13–28 Sketch the region enclosed by the given curves and find its area.

13. $y = 12 - x^2, \quad y = x^2 - 6$

14. $y = x^2, \quad y = 4x - x^2$

15. $y = \sec^2 x, \quad y = 8 \cos x, \quad -\pi/3 \leq x \leq \pi/3$

16. $y = \cos x, \quad y = 2 - \cos x, \quad 0 \leq x \leq 2\pi$

17. $x = 2y^2, \quad x = 4 + y^2$

18. $y = \sqrt{x - 1}, \quad x - y = 1$

19. $y = \cos \pi x, \quad y = 4x^2 - 1$

20. $x = y^4, \quad y = \sqrt{2 - x}, \quad y = 0$

21. $y = \cos x, \quad y = 1 - 2x/\pi$

22. $y = x^3, \quad y = x$

23. $y = \sqrt{2x}, \quad y = \frac{1}{3} x^2, \quad 0 \leq x \leq 6$

24. $y = \cos x, \quad y = 1 - \cos x, \quad 0 \leq x \leq \pi$

25. $y = x^4, \quad y = 2 - |x|$

26. $y = 3x - x^2, \quad y = x, \quad x = 3$

27. $y = 1/x^2, \quad y = x, \quad y = \frac{1}{5} x$

28. $y = \frac{1}{x^2}, \quad y = 2x^2, \quad x + y = 3, \quad x \geq 0$
29. The graphs of two functions are shown with the areas of the regions between the curves indicated. 
(a) What is the total area between the curves for $0 \leq x \leq 5$?
(b) What is the value of $\int_0^5 [f(x) - g(x)] \, dx$?

30–32 Sketch the region enclosed by the given curves and find its area.

30. $y = \frac{x}{\sqrt{1 + x^2}}$, $y = \frac{x}{\sqrt{9 - x^2}}$, $x \geq 0$

31. $y = \cos^2 x \sin x$, $y = \sin x$, $0 \leq x \leq \pi$

32. $y = x \sqrt{x^2 + 1}$, $y = x^3 \sqrt{x^2 + 1}$

33–34 Use calculus to find the area of the triangle with the given vertices.

33. (0, 0), (3, 1), (1, 2)
34. (2, 0), (0, 2), (−1, 1)

35–36 Evaluate the integral and interpret it as the area of a region. Sketch the region.

35. $\int_{-\pi/2}^{\pi/2} \left| \sin x - \cos 2x \right| \, dx$
36. $\int_0^4 \left| \sqrt{x + 2} - x \right| \, dx$

37–40 Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region bounded by the curves.

37. $y = x \sin(x^2)$, $y = x^4$, $x \geq 0$
38. $y = \frac{x}{(x^2 + 1)^2}$, $y = x^5 - x$, $x \geq 0$
39. $y = 3x^2 - 2x$, $y = x^3 - 3x + 4$
40. $y = x - \cos x$, $y = 2 - x^2$

41–44 Graph the region between the curves and use your calculator to compute the area correct to five decimal places.

41. $y = \frac{2}{1 + x^4}$, $y = x^2$
42. $y = x^6$, $y = \sqrt{2 - x^4}$
43. $y = \tan^2 x$, $y = \sqrt{x}$
44. $y = \cos x$, $y = x + 2 \sin x$

45. Use a computer algebra system to find the exact area enclosed by the curves $y = x^3 - 6x^3 + 4x$ and $y = x$.

46. Sketch the region in the $xy$-plane defined by the inequalities $x - 2y^2 \geq 0$, $1 - x - |y| \geq 0$ and find its area.

47. Racing cars driven by Chris and Kelly are side by side at the start of a race. The table shows the velocities of each car (in miles per hour) during the first ten seconds of the race. Use the Midpoint Rule to estimate how much farther Kelly travels than Chris does during the first ten seconds.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$v_C$</th>
<th>$v_K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>22</td>
</tr>
<tr>
<td>2</td>
<td>32</td>
<td>37</td>
</tr>
<tr>
<td>3</td>
<td>46</td>
<td>52</td>
</tr>
<tr>
<td>4</td>
<td>54</td>
<td>61</td>
</tr>
<tr>
<td>5</td>
<td>62</td>
<td>71</td>
</tr>
</tbody>
</table>

48. The widths (in meters) of a kidney-shaped swimming pool were measured at 2-meter intervals as indicated in the figure. Use the Midpoint Rule to estimate the area of the pool.

49. A cross-section of an airplane wing is shown. Measurements of the thickness of the wing, in centimeters, at 20-centimeter intervals are 5.8, 20.3, 26.7, 29.0, 27.6, 27.3, 23.8, 20.5, 15.1, 8.7, and 2.8. Use the Midpoint Rule to estimate the area of the wing’s cross-section.

50. If the birth rate of a population is $b(t) = 2200 + 52.3t + 0.74t^2$ people per year and the death rate is $d(t) = 1460 + 28.8t$ people per year, find the area between these curves for $0 \leq t \leq 10$. What does this area represent?

51. In Example 5, we modeled a measles pathogenesis curve by a function $f$. A patient infected with the measles virus
who has some immunity to the virus has a pathogenesis curve that can be modeled by, for instance, \( g(t) = 0.9f(t) \).

(a) If the same threshold concentration of the virus is required for infectiousness to begin as in Example 5, on what day does this occur?

(b) Let \( P_1 \) be the point on the graph of \( g \) where infectiousness begins. It has been shown that infectiousness ends at a point \( P_2 \) on the graph of \( g \) where the line through \( P_s, P_t \) has the same slope as the line through \( P_1, P_2 \) in Example 5(b). On what day does infectiousness end?

(c) Compute the level of infectiousness for this patient.

52. The rates at which rain fell, in inches per hour, in two different locations \( t \) hours after the start of a storm are given by \( f(t) = 0.73t^3 - 2t^2 + t + 0.6 \) and \( g(t) = 0.17t^2 - 0.5t + 1.1 \). Compute the area between the graphs for \( 0 \leq t \leq 2 \) and interpret your result in this context.

53. Two cars, A and B, start side by side and accelerate from rest. The figure shows the graphs of their velocity functions.

(a) Which car is ahead after one minute? Explain.

(b) What is the meaning of the area of the shaded region?

(c) Which car is ahead after two minutes? Explain.

(d) Estimate the time at which the cars are again side by side.

54. The figure shows graphs of the marginal revenue function \( R' \) and the marginal cost function \( C' \) for a manufacturer. [Recall from Section 3.7 that \( R(x) \) and \( C(x) \) represent the revenue and cost when \( x \) units are manufactured. Assume that \( R \) and \( C \) are measured in thousands of dollars.] What is the meaning of the area of the shaded region? Use the Midpoint Rule to estimate the value of this quantity.

55. The curve with equation \( y^2 = x^3 + 3 \) is called Tschirnhausen’s cubic. If you graph this curve you will see that part of the curve forms a loop. Find the area enclosed by the loop.

56. Find the area of the region bounded by the parabola \( y = x^2 \), the tangent line to this parabola at \((1, 1)\), and the \( x \)-axis.

57. Find the number \( b \) such that the line \( y = b \) divides the region bounded by the curves \( y = x^2 \) and \( y = 4 \) into two regions with equal area.

58. (a) Find the number \( a \) such that the line \( x = a \) bisects the area under the curve \( y = 1/x^2 \), \( 1 \leq x \leq 4 \).

(b) Find the number \( b \) such that the line \( y = b \) bisects the area in part (a).

59. Find the values of \( c \) such that the area of the region bounded by the parabolas \( y = x^2 - c^2 \) and \( y = c^2 - x^2 \) is 576.

60. Suppose that \( 0 < c < \pi/2 \). For what value of \( c \) is the area of the region enclosed by the curves \( y = \cos x \), \( y = \cos(x - c) \), and \( x = 0 \) equal to the area of the region enclosed by the curves \( y = \cos(x - c) \), \( x = \pi \), and \( y = 0 \)?

The following exercises are intended only for those who have already covered Chapter 6.

61–63 Sketch the region bounded by the given curves and find the area of the region.

61. \( y = 1/x, \ y = 1/x^2, \ x = 2 \)

62. \( y = \sin x, \ y = e^x, \ x = \pi/2 \)

63. \( y = \tan x, \ y = 2 \sin x, -\pi/3 \leq x \leq \pi/3 \)

64. For what values of \( m \) do the line \( y = mx \) and the curve \( y = x/(x^2 + 1) \) enclose a region? Find the area of the region.

**APPLIED PROJECT**

**THE GINI INDEX**

How is it possible to measure the distribution of income among the inhabitants of a given country? One such measure is the *Gini index*, named after the Italian economist Corrado Gini, who first devised it in 1912.

We first rank all households in a country by income and then we compute the percentage of households whose income is at most a given percentage of the country’s total income. We
define a Lorenz curve $y = L(x)$ on the interval $[0, 1]$ by plotting the point $(a/100, b/100)$ on the curve if the bottom $a\%$ of households receive at most $b\%$ of the total income. For instance, in Figure 1 the point $(0.4, 0.12)$ is on the Lorenz curve for the United States in 2010 because the poorest 40% of the population received just 12% of the total income. Likewise, the bottom 80% of the population received 50% of the total income, so the point $(0.8, 0.5)$ lies on the Lorenz curve. (The Lorenz curve is named after the American economist Max Lorenz.)

Figure 2 shows some typical Lorenz curves. They all pass through the points $(0, 0)$ and $(1, 1)$ and are concave upward. In the extreme case $L(x) = x$, society is perfectly egalitarian: the poorest $a\%$ of the population receives $a\%$ of the total income and so everybody receives the same income. The area between a Lorenz curve $y = L(x)$ and the line $y = x$ measures how much the income distribution differs from absolute equality. The Gini index (sometimes called the Gini coefficient or the coefficient of inequality) is the area between the Lorenz curve and the line $y = x$ (shaded in Figure 3) divided by the area under $y = x$.

1. (a) Show that the Gini index $G$ is twice the area between the Lorenz curve and the line $y = x$, that is,

$$G = 2 \int_0^1 [x - L(x)] \, dx$$

(b) What is the value of $G$ for a perfectly egalitarian society (everybody has the same income)? What is the value of $G$ for a perfectly totalitarian society (a single person receives all the income)?

2. The following table (derived from data supplied by the US Census Bureau) shows values of the Lorenz function for income distribution in the United States for the year 2010.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0.0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L(x)$</td>
<td>0.000</td>
<td>0.034</td>
<td>0.120</td>
<td>0.266</td>
<td>0.498</td>
<td>1.000</td>
</tr>
</tbody>
</table>

(a) What percentage of the total US income was received by the richest 20% of the population in 2010?
(b) Use a calculator or computer to fit a quadratic function to the data in the table. Graph the data points and the quadratic function. Is the quadratic model a reasonable fit?
(c) Use the quadratic model for the Lorenz function to estimate the Gini index for the United States in 2010.

3. The following table gives values for the Lorenz function in the years 1970, 1980, 1990, and 2000. Use the method of Problem 2 to estimate the Gini index for the United States for those years and compare with your answer to Problem 2(c). Do you notice a trend?

<table>
<thead>
<tr>
<th>$x$</th>
<th>0.0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1970</td>
<td>0.000</td>
<td>0.041</td>
<td>0.149</td>
<td>0.323</td>
<td>0.568</td>
<td>1.000</td>
</tr>
<tr>
<td>1980</td>
<td>0.000</td>
<td>0.042</td>
<td>0.144</td>
<td>0.312</td>
<td>0.559</td>
<td>1.000</td>
</tr>
<tr>
<td>1990</td>
<td>0.000</td>
<td>0.038</td>
<td>0.134</td>
<td>0.293</td>
<td>0.530</td>
<td>1.000</td>
</tr>
<tr>
<td>2000</td>
<td>0.000</td>
<td>0.036</td>
<td>0.125</td>
<td>0.273</td>
<td>0.503</td>
<td>1.000</td>
</tr>
</tbody>
</table>

4. A power model often provides a more accurate fit than a quadratic model for a Lorenz function. If you have a computer with Maple or Mathematica, fit a power function ($y = ax^k$) to the data in Problem 2 and use it to estimate the Gini index for the United States in 2010. Compare with your answer to parts (b) and (c) of Problem 2.
When we view the world around us, the light entering the eye near the center of the pupil is perceived brighter than light entering closer to the edges of the pupil. This phenomenon, known as the Stiles–Crawford effect, is explored as the pupil changes in radius in Exercise 90 on page 501.

THE COMMON THEME THAT LINKS the functions of this chapter is that they occur as pairs of inverse functions. In particular, two of the most important functions that occur in mathematics and its applications are the exponential function \( f(x) = b^x \) and its inverse function, the logarithmic function \( g(x) = \log_b x \). In this chapter we investigate their properties, compute their derivatives, and use them to describe exponential growth and decay in biology, physics, chemistry, and other sciences. We also study the inverses of trigonometric and hyperbolic functions. Finally, we look at a method (l’Hospital’s Rule) for computing difficult limits and apply it to sketching curves.

There are two possible ways of defining the exponential and logarithmic functions and developing their properties and derivatives. One is to start with the exponential function (defined as in algebra or precalculus courses) and then define the logarithm as its inverse. That is the approach taken in Sections 6.2, 6.3, and 6.4 and is probably the most intuitive method. The other way is to start by defining the logarithm as an integral and then define the exponential function as its inverse. This approach is followed in Sections 6.2*, 6.3*, and 6.4* and, although it is less intuitive, many instructors prefer it because it is more rigorous and the properties follow more easily. You need only read one of these two approaches (whichever your instructor recommends).
Table 1 gives data from an experiment in which a bacteria culture started with 100 bacteria in a limited nutrient medium; the size of the bacteria population was recorded at hourly intervals. The number of bacteria \( N \) is a function of the time \( t \): \( N = f(t) \).

Suppose, however, that the biologist changes her point of view and becomes interested in the time required for the population to reach various levels. In other words, she is thinking of \( t \) as a function of \( N \). This function is called the inverse function of \( f \), denoted \( f^{-1} \), and read “\( f \) inverse.” Thus \( t = f^{-1}(N) \) is the time required for the population level to reach \( N \). The values of \( f^{-1} \) can be found by reading Table 1 from right to left or by consulting Table 2. For instance, \( f^{-1}(550) = 6 \) because \( f(6) = 550 \).

**Table 1** \( N \) as a function of \( t \)

<table>
<thead>
<tr>
<th>( t ) (hours)</th>
<th>( N = f(t) ) = population at time ( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>1</td>
<td>168</td>
</tr>
<tr>
<td>2</td>
<td>259</td>
</tr>
<tr>
<td>3</td>
<td>358</td>
</tr>
<tr>
<td>4</td>
<td>445</td>
</tr>
<tr>
<td>5</td>
<td>509</td>
</tr>
<tr>
<td>6</td>
<td>550</td>
</tr>
<tr>
<td>7</td>
<td>573</td>
</tr>
<tr>
<td>8</td>
<td>586</td>
</tr>
</tbody>
</table>

**Table 2** \( t \) as a function of \( N \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( t = f^{-1}(N) ) = time to reach ( N ) bacteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>168</td>
<td>1</td>
</tr>
<tr>
<td>259</td>
<td>2</td>
</tr>
<tr>
<td>358</td>
<td>3</td>
</tr>
<tr>
<td>445</td>
<td>4</td>
</tr>
<tr>
<td>509</td>
<td>5</td>
</tr>
<tr>
<td>550</td>
<td>6</td>
</tr>
<tr>
<td>573</td>
<td>7</td>
</tr>
<tr>
<td>586</td>
<td>8</td>
</tr>
</tbody>
</table>

Not all functions possess inverses. Let’s compare the functions \( f \) and \( g \) whose arrow diagrams are shown in Figure 1. Note that \( f \) never takes on the same value twice (any two inputs in \( A \) have different outputs), whereas \( g \) does take on the same value twice (both 2 and 3 have the same output, 4). In symbols,

\[
g(2) = g(3)
\]

but

\[
f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2
\]

Functions that share this property with \( f \) are called one-to-one functions.

**Definition** A function \( f \) is called a one-to-one function if it never takes on the same value twice; that is,

\[
f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2
\]

If a horizontal line intersects the graph of \( f \) in more than one point, then we see from Figure 2 that there are numbers \( x_1 \) and \( x_2 \) such that \( f(x_1) = f(x_2) \). This means that \( f \) is not one-to-one. Therefore we have the following geometric method for determining whether a function is one-to-one.

**Horizontal Line Test** A function is one-to-one if and only if no horizontal line intersects its graph more than once.
EXAMPLE 1 Is the function \( f(x) = x^3 \) one-to-one?

SOLUTION 1 If \( x_1 \neq x_2 \), then \( x_1^3 \neq x_2^3 \) (two different numbers can’t have the same cube). Therefore, by Definition 1, \( f(x) = x^3 \) is one-to-one.

SOLUTION 2 From Figure 3 we see that no horizontal line intersects the graph of \( f(x) = x^3 \) more than once. Therefore, by the Horizontal Line Test, \( f \) is one-to-one.

EXAMPLE 2 Is the function \( g(x) = x^2 \) one-to-one?

SOLUTION 1 This function is not one-to-one because, for instance,

\[
g(1) = 1 = g(-1)
\]

and so 1 and \(-1\) have the same output.

SOLUTION 2 From Figure 4 we see that there are horizontal lines that intersect the graph of \( g \) more than once. Therefore, by the Horizontal Line Test, \( g \) is not one-to-one.

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

2 Definition Let \( f \) be a one-to-one function with domain \( A \) and range \( B \). Then its inverse function \( f^{-1} \) has domain \( B \) and range \( A \) and is defined by

\[
f^{-1}(y) = x \iff f(x) = y
\]

for any \( y \) in \( B \).

This definition says that if \( f \) maps \( x \) into \( y \), then \( f^{-1} \) maps \( y \) back into \( x \). (If \( f \) were not one-to-one, then \( f^{-1} \) would not be uniquely defined.) The arrow diagram in Figure 5 indicates that \( f^{-1} \) reverses the effect of \( f \). Note that

\[
\text{domain of } f^{-1} = \text{range of } f
\]

\[
\text{range of } f^{-1} = \text{domain of } f
\]

For example, the inverse function of \( f(x) = x^3 \) is \( f^{-1}(x) = x^{1/3} \) because if \( y = x^3 \), then

\[
f^{-1}(y) = f^{-1}(x^3) = (x^3)^{1/3} = x
\]

CAUTION Do not mistake the \(-1\) in \( f^{-1} \) for an exponent. Thus

\[
f^{-1}(x) \text{ does not mean } \frac{1}{f(x)}
\]

The reciprocal \( 1/f(x) \) could, however, be written as \( [f(x)]^{-1} \).
EXAMPLE 3 If \( f(1) = 5, f(3) = 7, \) and \( f(8) = -10 \), find \( f^{-1}(7), f^{-1}(5), \) and \( f^{-1}(-10) \).

**SOLUTION** From the definition of \( f^{-1} \) we have

\[
\begin{align*}
f^{-1}(7) &= 3 & \text{because} & \quad f(3) &= 7 \\
f^{-1}(5) &= 1 & \text{because} & \quad f(1) &= 5 \\
f^{-1}(-10) &= 8 & \text{because} & \quad f(8) &= -10
\end{align*}
\]

The diagram in Figure 6 makes it clear how \( f^{-1} \) reverses the effect of \( f \) in this case.

The letter \( x \) is traditionally used as the independent variable, so when we concentrate on \( f^{-1} \) rather than on \( f \), we usually reverse the roles of \( x \) and \( y \) in Definition 2 and write

\[
\begin{align*}
f^{-1}(x) &= y & \iff & \quad f(y) &= x
\end{align*}
\]

By substituting for \( y \) in Definition 2 and substituting for \( x \) in (3), we get the following cancellation equations:

\[
\begin{align*}
f^{-1}(f(x)) &= x & \text{for every} & \quad x \in A \\
f(f^{-1}(x)) &= x & \text{for every} & \quad x \in B
\end{align*}
\]

The first cancellation equation says that if we start with \( x \), apply \( f \), and then apply \( f^{-1} \), we arrive back at \( x \), where we started (see the machine diagram in Figure 7). Thus \( f^{-1} \) undoes what \( f \) does. The second equation says that \( f \) undoes what \( f^{-1} \) does.

For example, if \( f(x) = x^3 \), then \( f^{-1}(x) = x^{1/3} \) and so the cancellation equations become

\[
\begin{align*}
f^{-1}(f(x)) &= (x^3)^{1/3} = x \\
f(f^{-1}(x)) &= (x^{1/3})^3 = x
\end{align*}
\]

These equations simply say that the cube function and the cube root function cancel each other when applied in succession. Now let’s see how to compute inverse functions. If we have a function \( y = f(x) \) and are able to solve this equation for \( x \) in terms of \( y \), then according to Definition 2 we must have \( x = f^{-1}(y) \). If we want to call the independent variable \( x \), we then interchange \( x \) and \( y \) and arrive at the equation \( y = f^{-1}(x) \).
How to Find the Inverse Function of a One-to-One Function $f$

**STEP 1** Write $y = f(x)$.

**STEP 2** Solve this equation for $x$ in terms of $y$ (if possible).

**STEP 3** To express $f^{-1}$ as a function of $x$, interchange $x$ and $y$. The resulting equation is $y = f^{-1}(x)$.

**EXAMPLE 4** Find the inverse function of $f(x) = x^3 + 2$.

**SOLUTION** According to (5) we first write

$$y = x^3 + 2$$

Then we solve this equation for $x$:

$$x^3 = y - 2$$

$$x = \sqrt[3]{y - 2}$$

Finally, we interchange $x$ and $y$:

$$y = \sqrt[3]{x - 2}$$

Therefore the inverse function is $f^{-1}(x) = \sqrt[3]{x - 2}$. ■

In Example 4, notice how $f^{-1}$ reverses the effect of $f$. The function $f$ is the rule “Cube, then add 2”; $f^{-1}$ is the rule “Subtract 2, then take the cube root.”

The principle of interchanging $x$ and $y$ to find the inverse function also gives us the method for obtaining the graph of $f^{-1}$ from the graph of $f$. Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point $(a, b)$ is on the graph of $f$ if and only if the point $(b, a)$ is on the graph of $f^{-1}$. But we get the point $(b, a)$ from $(a, b)$ by reflecting about the line $y = x$. (See Figure 8.)

The graph of $f^{-1}$ is obtained by reflecting the graph of $f$ about the line $y = x$. 

**FIGURE 8**

**FIGURE 9**

Therefore, as illustrated by Figure 9:
EXAMPLE 5 Sketch the graphs of \( f(x) = \sqrt{-1 - x} \) and its inverse function using the same coordinate axes.

SOLUTION First we sketch the curve \( y = \sqrt{-1 - x} \) (the top half of the parabola \( y^2 = -1 - x \), or \( x = -y^2 - 1 \)) and then we reflect about the line \( y = x \) to get the graph of \( f^{-1} \). (See Figure 10.) As a check on our graph, notice that the expression for \( f^{-1} \) is \( f^{-1}(x) = -x^2 - 1, x \geq 0 \). So the graph of \( f^{-1} \) is the right half of the parabola \( y = -x^2 - 1 \) and this seems reasonable from Figure 10.

The Calculus of Inverse Functions

Now let’s look at inverse functions from the point of view of calculus. Suppose that \( f \) is both one-to-one and continuous. We think of a continuous function as one whose graph has no break in it. (It consists of just one piece.) Since the graph of \( f \) has no break in it either. We therefore expect that \( f^{-1} \) has no corner or kink in it either (see Figure 9). Thus we might expect that \( f^{-1} \) is also a continuous function.

This geometrical argument does not prove the following theorem but at least it makes the theorem plausible. A proof can be found in Appendix F.

6 Theorem If \( f \) is a one-to-one continuous function defined on an interval, then its inverse function \( f^{-1} \) is also continuous.

Now suppose that \( f \) is a one-to-one differentiable function. Geometrically we can think of a differentiable function as one whose graph has no corner or kink in it. We get the graph of \( f^{-1} \) by reflecting the graph of \( f \) about the line \( y = x \), so the graph of \( f^{-1} \) has no corner or kink in it either. We therefore expect that \( f^{-1} \) is also differentiable (except where its tangents are vertical). In fact, we can predict the value of the derivative of \( f^{-1} \) at a given point by a geometric argument. In Figure 11 the graphs of \( f \) and its inverse \( f^{-1} \) are shown. If \( f(b) = a \), then \( f^{-1}(a) = b \) and \( (f^{-1})'(a) \) is the slope of the tangent line \( L \) to the graph of \( f^{-1} \) at \( (a, b) \), which is \( \Delta y/\Delta x \). Reflecting in the line \( y = x \) has the effect of interchanging the \( x \)- and \( y \)-coordinates. So the slope of the reflected line \( \ell \) [the tangent to the graph of \( f \) at \( (b, a) \)] is \( \Delta x/\Delta y \). Thus the slope of \( L \) is the reciprocal of the slope of \( \ell \), that is,

\[
(f^{-1})'(a) = \frac{\Delta y}{\Delta x} = \frac{1}{\Delta x/\Delta y} = \frac{1}{f'(b)}
\]

7 Theorem If \( f \) is a one-to-one differentiable function with inverse function \( f^{-1} \) and \( f'(f^{-1}(a)) \neq 0 \), then the inverse function is differentiable at \( a \) and

\[
(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}
\]

PROOF Write the definition of derivative as in Equation 2.1.5:

\[
(f^{-1})'(a) = \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a}
\]

If \( f(b) = a \), then \( f^{-1}(a) = b \). And if we let \( y = f^{-1}(x) \), then \( f(y) = x \). Since \( f \) is differentiable, it is continuous, so \( f^{-1} \) is continuous by Theorem 6. Thus if \( x \to a \),

\[
(f^{-1})'(a) = \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a}
\]
then \( f^{-1}(x) \to f^{-1}(a) \), that is, \( y \to b \). Therefore

\[
(f^{-1})'(a) = \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a} = \lim_{y \to b} \frac{y - b}{f(y) - f(b)}
\]

\[
= \lim_{y \to b} \frac{1}{f(y) - f(b)} \cdot \frac{f(y) - f(b)}{y - b} = \lim_{y \to b} \frac{1}{y - b} \cdot \frac{1}{f'(f^{-1}(a))}
\]

\[
= \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}
\]

**NOTE 1** Replacing \( a \) by the general number \( x \) in the formula of Theorem 7, we get

\[
(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}
\]

If we write \( y = f^{-1}(x) \), then \( f(y) = x \), so Equation 8, when expressed in Leibniz notation, becomes

\[
\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}
\]

**NOTE 2** If it is known in advance that \( f^{-1} \) is differentiable, then its derivative can be computed more easily than in the proof of Theorem 7 by using implicit differentiation. If \( y = f^{-1}(x) \), then \( f(y) = x \). Differentiating the equation \( f(y) = x \) implicitly with respect to \( x \), remembering that \( y \) is a function of \( x \), and using the Chain Rule, we get

\[
f'(y) \frac{dy}{dx} = 1
\]

Therefore

\[
\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{\frac{dx}{dy}}
\]

**EXAMPLE 6** Although the function \( y = x^2 \), \( x \in \mathbb{R} \), is not one-to-one and therefore does not have an inverse function, we can turn it into a one-to-one function by restricting its domain. For instance, the function \( f(x) = x^2 \), \( 0 \leq x \leq 2 \), is one-to-one (by the Horizontal Line Test) and has domain \([0, 2]\) and range \([0, 4]\). (See Figure 12.) Thus \( f \) has an inverse function \( f^{-1} \) with domain \([0, 4]\) and range \([0, 2]\).
Without computing a formula for \((f^{-1})'\) we can still calculate \((f^{-1})'(1)\). Since 
\[ f(1) = 1, \text{ we have } f^{-1}(1) = 1. \] Also \(f'(x) = 2x\). So by Theorem 7 we have 
\[
(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(1)} = \frac{1}{2}
\]

In this case it is easy to find \(f^{-1}\) explicitly. In fact, 
\[ f^{-1}(x) = \sqrt{x}, \ 0 \leq x \leq 4. \] In general, we could use the method given by (5). Then \((f^{-1})'(x) = 1/(2\sqrt{x})\), so \((f^{-1})'(1) = \frac{1}{2}\), which agrees with the preceding computation. The functions \(f\) and \(f^{-1}\) are graphed in Figure 13.

**EXAMPLE 7** If \(f(x) = 2x + \cos x\), find \((f^{-1})'(1)\).

**SOLUTION** Notice that \(f\) is one-to-one because 
\[ f'(x) = 2 - \sin x > 0 \]

and so \(f\) is increasing. To use Theorem 7 we need to know \(f^{-1}(1)\) and we can find it by inspection:
\[
 f(0) = 1 \Rightarrow f^{-1}(1) = 0
\]

Therefore 
\[
(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{2 - \sin 0} = \frac{1}{2}
\]

### 6.1 EXERCISES

1. (a) What is a one-to-one function?
   (b) How can you tell from the graph of a function whether it is one-to-one?

2. (a) Suppose \(f\) is a one-to-one function with domain \(A\) and range \(B\). How is the inverse function \(f^{-1}\) defined? What is the domain of \(f^{-1}\)? What is the range of \(f^{-1}\)?
   (b) If you are given a formula for \(f\), how do you find a formula for \(f^{-1}\)?
   (c) If you are given the graph of \(f\), how do you find the graph of \(f^{-1}\)?

3–16 A function is given by a table of values, a graph, a formula, or a verbal description. Determine whether it is one-to-one.

<table>
<thead>
<tr>
<th>(x)</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>1.5</td>
<td>2.0</td>
<td>3.6</td>
<td>5.3</td>
<td>2.8</td>
<td>2.0</td>
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</tbody>
</table>

4. | \(x\) |   |   |   |   |   |   |
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>1.0</td>
<td>1.9</td>
<td>2.8</td>
<td>3.5</td>
<td>3.1</td>
<td>2.9</td>
</tr>
</tbody>
</table>

5. [Graph of a function]

6. [Graph of a function]

<table>
<thead>
<tr>
<th>(x)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7. [Graph of a function]

<table>
<thead>
<tr>
<th>(y)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td></td>
</tr>
</tbody>
</table>

8. [Graph of a function]

<table>
<thead>
<tr>
<th>(y)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td></td>
</tr>
</tbody>
</table>

9. \(f(x) = 2x - 3\)
10. \(f(x) = x^4 - 16\)
11. \(g(x) = 1 - \sin x\)
12. \(g(x) = \sqrt{x}\)
13. \(h(x) = 1 + \cos x\)
14. \(h(x) = 1 + \cos x, \ 0 \leq x \leq \pi\)
15. \(f(t)\) is the height of a football \(t\) seconds after kickoff.
16. \(f(t)\) is your height at age \(t\).

17. Assume that \(f\) is a one-to-one function.
   (a) If \(f(6) = 17\), what is \(f^{-1}(17)\)?
   (b) If \(f^{-1}(3) = 2\), what is \(f(2)\)?

18. If \(f(x) = x^3 + x^2\), find \(f^{-1}(3)\) and \(f(f^{-1}(2))\).
19. If \(h(x) = x + \sqrt{x}\), find \(h^{-1}(6)\).

20. The graph of \(f\) is given.
   (a) Why is \(f\) one-to-one?
   (b) What are the domain and range of \(f^{-1}\)?
(c) What is the value of \( f^{-1}(2) \)?

(d) Estimate the value of \( f^{-1}(0) \).

21. The formula \( C = \frac{5}{9}(F - 32) \), where \( F \geq -459.67 \), expresses the Celsius temperature \( C \) as a function of the Fahrenheit temperature \( F \). Find a formula for the inverse function and interpret it. What is the domain of the inverse function?

22. In the theory of relativity, the mass of a particle with speed \( v \) is

\[
m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}
\]

where \( m_0 \) is the rest mass of the particle and \( c \) is the speed of light in a vacuum. Find the inverse function of \( f \) and explain its meaning.

23–28 Find a formula for the inverse of the function.

23. \( f(x) = 5 - 4x \)

24. \( f(x) = \frac{4x - 1}{2x + 3} \)

25. \( f(x) = 1 + \sqrt{2 + 3x} \)

26. \( y = x^2 - x, \ x \geq \frac{1}{2} \)

27. \( y = \frac{1 - \sqrt{x}}{1 + \sqrt{x}} \)

28. \( f(x) = 2x^2 - 8x, \ x \geq 2 \)

29–30 Find an explicit formula for \( f^{-1} \) and use it to graph \( f^{-1}, f \), and the line \( y = x \) on the same screen. To check your work, see whether the graphs of \( f \) and \( f^{-1} \) are reflections about the line.

29. \( f(x) = \sqrt{4x + 3} \)

30. \( f(x) = 2 - x^4, \ x \geq 0 \)

31–32 Use the given graph of \( f \) to sketch the graph of \( f^{-1} \).

31.

32.

33. Let \( f(x) = \sqrt{1 - x^2}, \ 0 \leq x \leq 1 \).

(a) Find \( f^{-1} \). How is it related to \( f \)?

(b) Identify the graph of \( f \) and explain your answer to part (a).

34. Let \( g(x) = \sqrt{1 - x^3} \).

(a) Find \( g^{-1} \). How is it related to \( g \)?

(b) Graph \( g \). How do you explain your answer to part (a)?

35–38

(a) Show that \( f \) is one-to-one.

(b) Use Theorem 7 to find \( (f^{-1})'(a) \).

(c) Calculate \( f^{-1}(x) \) and state the domain and range of \( f^{-1} \).

(d) Calculate \( (f^{-1})'(a) \) from the formula in part (c) and check that it agrees with the result of part (b).

(e) Sketch the graphs of \( f \) and \( f^{-1} \) on the same axes.

35. \( f(x) = x^3, \ a = 8 \)

36. \( f(x) = \sqrt{x - 2}, \ a = 2 \)

37. \( f(x) = 9 - x^2, \ 0 \leq x \leq 3, \ a = 8 \)

38. \( f(x) = 1/(x - 1), \ x > 1, \ a = 2 \)

39–42 Find \( (f^{-1})'(a) \).

39. \( f(x) = 3x^3 + 4x^2 + 6x + 5, \ a = 5 \)

40. \( f(x) = x^3 + 3 \sin x + 2 \cos x, \ a = 2 \)

41. \( f(x) = 3 + x^2 + \tan(\pi x/2), \ -1 < x < 1, \ a = 3 \)

42. \( f(x) = \sqrt{x^3 + 4x + 4}, \ a = 3 \)

43. Suppose \( f^{-1} \) is the inverse function of a differentiable function \( f \) and \( f(4) = 5, f'(4) = \frac{1}{2} \). Find \( (f^{-1})'(5) \).

44. If \( g \) is an increasing function such that \( g(2) = 8 \) and \( g'(2) = 5 \), calculate \( (g^{-1})'(8) \).

45. If \( f(x) = \int_1^x \sqrt{1 + t^2} \, dt \), find \( (f^{-1})'(0) \).

46. Suppose \( f^{-1} \) is the inverse function of a differentiable function \( f \) and let \( G(x) = 1/f^{-1}(x) \). If \( f(3) = 2 \) and \( f'(3) = \frac{1}{9} \), find \( G'(2) \).

47. Graph the function \( f(x) = \sqrt{x^3 + x^2 + x + 1} \) and explain why it is one-to-one. Then use a computer algebra system to find an explicit expression for \( f^{-1}(x) \). (Your CAS will produce three possible expressions. Explain why two of them are irrelevant in this context.)

48. Show that \( h(x) = \sin x, \ x \in \mathbb{R} \), is not one-to-one, but its restriction \( f(x) = \sin x, -\pi/2 \leq x \leq \pi/2 \), is one-to-one. Compute the derivative of \( f^{-1} = \sin^{-1} \) by the method of Note 2.

49. (a) If we shift a curve to the left, what happens to its reflection about the line \( y = x \)? In view of this geometric principle, find an expression for the inverse of \( g(x) = \sqrt{1 - x^2}, \) where \( f \) is a one-to-one function.

(b) Find an expression for the inverse of \( h(x) = f(cx) \), where \( c \neq 0 \).

50. (a) If \( f \) is a one-to-one, twice differentiable function with inverse function \( g \), show that

\[
g''(x) = \frac{-f''(g(x))}{[f'(g(x))]^3}
\]

(b) Deduce that if \( f \) is increasing and concave upward, then its inverse function is concave downward.
The function \( f(x) = 2^x \) is called an exponential function because the variable, \( x \), is the exponent. It should not be confused with the power function \( g(x) = x^2 \), in which the variable is the base.

In general, an exponential function is a function of the form

\[
  f(x) = b^x
\]

where \( b \) is a positive constant. Let’s recall what this means.

If \( x = n \), a positive integer, then

\[
  b^n = b \cdot b \cdot \cdots \cdot b
\]

\( n \) factors

If \( x = 0 \), then \( b^0 = 1 \), and if \( x = -n \), where \( n \) is a positive integer, then

\[
  b^{-n} = \frac{1}{b^n}
\]

If \( x \) is a rational number, \( x = p/q \), where \( p \) and \( q \) are integers and \( q > 0 \), then

\[
  b^x = b^{p/q} = \sqrt[q]{b^p} = \left(\sqrt[q]{b}\right)^p
\]

But what is the meaning of \( b^x \) if \( x \) is an irrational number? For instance, what is meant by \( 2^{\sqrt{3}} \) or \( 5^{\pi} \)?

To help us answer this question we first look at the graph of the function \( y = 2^x \), where \( x \) is rational. A representation of this graph is shown in Figure 1. We want to enlarge the domain of \( y = 2^x \) to include both rational and irrational numbers.

There are holes in the graph in Figure 1 corresponding to irrational values of \( x \). We want to fill in the holes by defining \( f(x) = 2^x \), where \( x \in \mathbb{R} \), so that \( f \) is an increasing function. In particular, since the irrational number \( \sqrt{3} \) satisfies

\[
  1.7 < \sqrt{3} < 1.8
\]

we must have

\[
  2^{1.7} < 2^{\sqrt{3}} < 2^{1.8}
\]

and we know what \( 2^{1.7} \) and \( 2^{1.8} \) mean because 1.7 and 1.8 are rational numbers. Similarly, if we use better approximations for \( \sqrt{3} \), we obtain better approximations for \( 2^{\sqrt{3}} \):

\[
  1.73 < \sqrt{3} < 1.74 \quad \Rightarrow \quad 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74}
\]

\[
  1.732 < \sqrt{3} < 1.733 \quad \Rightarrow \quad 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733}
\]

\[
  1.7320 < \sqrt{3} < 1.7321 \quad \Rightarrow \quad 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321}
\]

\[
  1.73205 < \sqrt{3} < 1.73206 \quad \Rightarrow \quad 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206}
\]

\[
  \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\]

It can be shown that there is exactly one number that is greater than all of the numbers

\[
  2^{1.7}, \quad 2^{1.73}, \quad 2^{1.732}, \quad 2^{1.7320}, \quad 2^{1.73205}, \quad \ldots
\]

and less than all of the numbers

\[
  2^{1.8}, \quad 2^{1.74}, \quad 2^{1.733}, \quad 2^{1.7321}, \quad 2^{1.73206}, \quad \ldots
\]

We define \( 2^{\sqrt{3}} \) to be this number. Using the preceding approximation process we can...
compute it correct to six decimal places:

$$2^{\sqrt{3}} = 3.321997$$

Similarly, we can define $2^x$ (or $b^x$, if $b > 0$) where $x$ is any irrational number. Figure 2 shows how all the holes in Figure 1 have been filled to complete the graph of the function $f(x) = 2^x$, $x \in \mathbb{R}$.

In general, if $b$ is any positive number, we define

$$b^x = \lim_{r \to x} b^r \quad r \text{ rational}$$

This definition makes sense because any irrational number can be approximated as closely as we like by a rational number. For instance, because $\sqrt{3}$ has the decimal representation $\sqrt{3} = 1.7320508 \ldots$, Definition 1 says that $2^{\sqrt{3}}$ is the limit of the sequence of numbers

$$2^{1.7}, \quad 2^{1.73}, \quad 2^{1.732}, \quad 2^{1.7320}, \quad 2^{1.73205}, \quad 2^{1.732050}, \quad 2^{1.7320508}, \ldots$$

Similarly, $5^x$ is the limit of the sequence of numbers

$$5^{3.1}, \quad 5^{3.14}, \quad 5^{3.141}, \quad 5^{3.1415}, \quad 5^{3.14159}, \quad 5^{3.141592}, \quad 5^{3.1415926}, \ldots$$

It can be shown that Definition 1 uniquely specifies $b^x$ and makes the function $f(x) = b^x$ continuous.

The graphs of members of the family of functions $y = b^x$ are shown in Figure 3 for various values of the base $b$. Notice that all of these graphs pass through the same point $(0, 1)$ because $b^0 = 1$ for $b \neq 0$. Notice also that as the base $b$ gets larger, the exponential function grows more rapidly (for $x > 0$).

Figure 4 shows how the exponential function $y = 2^x$ compares with the power function $y = x^2$. The graphs intersect three times, but ultimately the exponential curve $y = 2^x$ grows far more rapidly than the parabola $y = x^2$. (See also Figure 5.)

You can see from Figure 3 that there are basically three kinds of exponential functions $y = b^x$. If $0 < b < 1$, the exponential function decreases; if $b = 1$, it is a constant; and if $b > 1$, it increases. These three cases are illustrated in Figure 6. Because
Chapter 6  Inverse Functions

(1/b)^x = 1/(b^x) = b^-x, the graph of y = (1/b)^x is just the reflection of the graph of y = b^x about the y-axis.

![Figure 6](http://www.stewartcalculus.com)

(a) y = b^x, 0 < b < 1  
(b) y = 1^x  
(c) y = b^x, b > 1

The properties of the exponential function are summarized in the following theorem.

2. **Theorem**  If b > 0 and b ≠ 1, then f(x) = b^x is a continuous function with domain \( \mathbb{R} \) and range (0, ∞). In particular, b^x > 0 for all x. If 0 < b < 1, f(x) = b^x is a decreasing function; if b > 1, f is an increasing function. If a, b > 0 and x, y ∈ \( \mathbb{R} \), then

1. \( b^{x+y} = b^x b^y \)  
2. \( b^{x-y} = \frac{b^x}{b^y} \)  
3. \( (b^x)^y = b^{xy} \)  
4. \( (ab)^x = a^x b^x \)

The reason for the importance of the exponential function lies in properties 1–4, which are called the **Laws of Exponents**. If x and y are rational numbers, then these laws are well known from elementary algebra. For arbitrary real numbers x and y these laws can be deduced from the special case where the exponents are rational by using Equation 1.

The following limits can be read from the graphs shown in Figure 6 or proved from the definition of a limit at infinity. (See Exercise 6.3.71.)

3. If b > 1, then \( \lim_{x \to -\infty} b^x = 0 \) and \( \lim_{x \to \infty} b^x = \infty \)

If 0 < b < 1, then \( \lim_{x \to -\infty} b^x = \infty \) and \( \lim_{x \to \infty} b^x = 0 \)

In particular, if b ≠ 1, then the x-axis is a horizontal asymptote of the graph of the exponential function y = b^x.

**EXAMPLE 1**

(a) Find \( \lim_{x \to -\infty} (2^{-x} - 1) \).
(b) Sketch the graph of the function y = 2^{-x} − 1.

**SOLUTION**

(a) \( \lim_{x \to -\infty} (2^{-x} - 1) = \lim_{x \to -\infty} \left[ \left( \frac{1}{2} \right)^x - 1 \right] \)

\[ = 0 - 1 \quad \text{[by (3) with } b = \frac{1}{2} < 1] \]

\[ = -1 \]

(b) We write \( y = \left( \frac{1}{2} \right)^x - 1 \) as in part (a). The graph of \( y = \left( \frac{1}{2} \right)^x \) is shown in Figure 3, so we shift it down one unit to obtain the graph of \( y = \left( \frac{1}{2} \right)^x - 1 \) shown in Figure 7. (For a review of shifting graphs, see Section 1.3.) Part (a) shows that the line y = −1 is a horizontal asymptote.
Applications of Exponential Functions

The exponential function occurs very frequently in mathematical models of nature and society. Here we indicate briefly how it arises in the description of population growth. In Section 6.5 we will pursue these and other applications in greater detail.

In Section 2.7 we considered a bacteria population that doubles every hour and saw that if the initial population is $n_0$, then the population after $t$ hours is given by the function $f(t) = n_02^t$. This population function is a constant multiple of the exponential function $y = 2^t$, so it exhibits the rapid growth that we observed in Figures 2 and 5. Under ideal conditions (unlimited space and nutrition and absence of disease) this exponential growth is typical of what actually occurs in nature.

What about the human population? Table 1 shows data for the population of the world in the 20th century and Figure 8 shows the corresponding scatter plot.

The pattern of the data points in Figure 8 suggests exponential growth, so we use a graphing calculator with exponential regression capability to apply the method of least squares and obtain the exponential model

$$P = (1436.53) \cdot (1.01395)^t$$

where $t = 0$ corresponds to the year 1900. Figure 9 shows the graph of this exponential function together with the original data points. We see that the exponential curve fits the data reasonably well. The period of relatively slow population growth is explained by the two world wars and the Great Depression of the 1930s.
In 1995 a paper appeared detailing the effect of the protease inhibitor ABT-538 on the human immunodeficiency virus HIV-1. Table 2 shows values of the plasma viral load \( V(t) \) of patient 303, measured in RNA copies per mL, \( t \) days after ABT-538 treatment was begun. The corresponding scatter plot is shown in Figure 10.

<table>
<thead>
<tr>
<th>( t ) (days)</th>
<th>( V(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>76.0</td>
</tr>
<tr>
<td>4</td>
<td>53.0</td>
</tr>
<tr>
<td>8</td>
<td>18.0</td>
</tr>
<tr>
<td>11</td>
<td>9.4</td>
</tr>
<tr>
<td>15</td>
<td>5.2</td>
</tr>
<tr>
<td>22</td>
<td>3.6</td>
</tr>
</tbody>
</table>

The rather dramatic decline of the viral load that we see in Figure 10 reminds us of the graphs of the exponential function \( y = b^x \) in Figures 3 and 6(a) for the case where the base \( b \) is less than 1. So let’s model the function \( V(t) \) by an exponential function. Using a graphing calculator or computer to fit the data in Table 2 with an exponential function of the form \( y = a \cdot b^x \), we obtain the model

\[
V = 96.39785 \cdot (0.818656)^t
\]

In Figure 11 we graph this exponential function with the data points and see that the model represents the viral load reasonably well for the first month of treatment.

We could use the graph in Figure 11 to estimate the half-life of \( V \), that is, the time required for the viral load to be reduced to half its initial value (see Exercise 63).

### Derivatives of Exponential Functions

Let’s try to compute the derivative of the exponential function \( f(x) = b^x \) using the definition of a derivative:

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h}
\]

\[
= \lim_{h \to 0} b^x \frac{b^h - 1}{h} = \lim_{h \to 0} \frac{b^x(b^h - 1)}{h}
\]

---

The factor \( b^x \) doesn’t depend on \( h \), so we can take it in front of the limit:

\[
f'(x) = b^x \lim_{h \to 0} \frac{b^h - 1}{h}
\]

Notice that the limit is the value of the derivative of \( f \) at 0, that is,

\[
\lim_{h \to 0} \frac{b^h - 1}{h} = f'(0)
\]

Therefore we have shown that if the exponential function \( f(x) = b^x \) is differentiable at 0, then it is differentiable everywhere and

\[
f'(x) = f'(0)b^x
\]

This equation says that the rate of change of any exponential function is proportional to the function itself. (The slope is proportional to the height.)

Numerical evidence for the existence of \( f'(0) \) is given in the table at the left for the cases \( b = 2 \) and \( b = 3 \). (Values are stated correct to four decimal places.) It appears that the limits exist and

- for \( b = 2 \), \( f'(0) = \lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.69 \)
- for \( b = 3 \), \( f'(0) = \lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.10 \)

In fact, it can be proved that these limits exist and, correct to six decimal places, the values are

\[
\frac{d}{dx} (2^x) \bigg|_{x=0} = 0.693147 \\
\frac{d}{dx} (3^x) \bigg|_{x=0} = 1.098612
\]

Thus, from Equation 4, we have

\[
\frac{d}{dx} (2^x) = (0.69)2^x \\
\frac{d}{dx} (3^x) = (1.10)3^x
\]

Of all possible choices for the base \( b \) in Equation 4, the simplest differentiation formula occurs when \( f'(0) = 1 \). In view of the estimates of \( f'(0) \) for \( b = 2 \) and \( b = 3 \), it seems reasonable that there is a number \( b \) between 2 and 3 for which \( f'(0) = 1 \). It is traditional to denote this value by the letter \( e \). Thus we have the following definition.

**Definition of the Number \( e \)**

\[ e \] is the number such that \( \lim_{h \to 0} \frac{e^h - 1}{h} = 1 \)

Geometrically, this means that of all the possible exponential functions \( y = b^x \), the function \( f(x) = e^x \) is the one whose tangent line at \((0, 1)\) has a slope \( f'(0) \) that is exactly 1. (See Figures 12 and 13.) We call the function \( f(x) = e^x \) the natural exponential function.
If we put \( b = e \) and, therefore, \( f'(0) = 1 \) in Equation 4, it becomes the following important differentiation formula.

### Derivative of the Natural Exponential Function

\[
\frac{d}{dx}(e^x) = e^x
\]

Thus the exponential function \( f(x) = e^x \) has the property that it is its own derivative. The geometrical significance of this fact is that the slope of a tangent line to the curve \( y = e^x \) is equal to the \( y \)-coordinate of the point (see Figure 13).

**EXAMPLE 2** Differentiate the function \( y = e^{\tan x} \).

**SOLUTION** To use the Chain Rule, we let \( u = \tan x \). Then we have \( y = e^u \), so

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = e^u \frac{du}{dx} = e^{\tan x} \sec^2 x
\]

In general if we combine Formula 8 with the Chain Rule, as in Example 2, we get

\[
\frac{d}{dx}(e^u) = e^u \frac{du}{dx}
\]

**EXAMPLE 3** Find \( y' \) if \( y = e^{-4x} \sin 5x \).

**SOLUTION** Using Formula 9 and the Product Rule, we have

\[
y' = e^{-4x} (\cos 5x)(5) + (\sin 5x)e^{-4x}(-4) = e^{-4x}(5 \cos 5x - 4 \sin 5x)
\]

We have seen that \( e \) is a number that lies somewhere between 2 and 3, but we can use Equation 4 to estimate the numerical value of \( e \) more accurately. Let \( e = 2^\varepsilon \). Then \( e^\varepsilon = 2^\varepsilon \). If \( f(x) = 2^x \), then from Equation 4 we have \( f'(x) = k2^x \), where the value of \( k \)
is \( f'(0) \approx 0.693147 \) (see Equations 5). Thus, by the Chain Rule,

\[
e^x = \frac{d}{dx} (e^x) = \frac{d}{dx} (2^c x) = k2^c x \frac{d}{dx} (cx) = ck2^c x
\]

Putting \( x = 0 \), we have \( 1 = ck \), so \( c = 1/k \) and

\[
e = 2^{1/k} = 2^{1/0.693147} \approx 2.71828
\]

It can be shown that the approximate value to 20 decimal places is

\[
e \approx 2.71828182845904523536
\]

The decimal expansion of \( e \) is nonrepeating because \( e \) is an irrational number.

**EXAMPLE 4** In Example 2.7.6 we considered a population of bacteria cells in a homogeneous nutrient medium. We showed that if the population doubles every hour, then the population after \( t \) hours is

\[
n = n_0 2^t
\]

where \( n_0 \) is the initial population. Now we can use (4) and (5) to compute the growth rate:

\[
\frac{dn}{dt} = n_0 (0.693147) 2^t
\]

For instance, if the initial population is \( n_0 = 1000 \) cells, then the growth rate after two hours is

\[
\left. \frac{dn}{dt} \right|_{t=2} \approx (1000)(0.693147)2^t |_{t=2} = (4000)(0.693147) \approx 2773 \text{ cells/h}
\]

**EXAMPLE 5** Find the absolute maximum value of the function \( f(x) = xe^{-x} \).

**SOLUTION** We differentiate to find any critical numbers:

\[
f'(x) = xe^{-x}(-1) + e^{-x}(1) = e^{-x}(1 - x)
\]

Since exponential functions are always positive, we see that \( f'(x) > 0 \) when \( 1 - x > 0 \), that is, when \( x < 1 \). Similarly, \( f'(x) < 0 \) when \( x > 1 \). By the First Derivative Test for Absolute Extreme Values, \( f \) has an absolute maximum value when \( x = 1 \) and the value is

\[
f(1) = (1)e^{-1} = \frac{1}{e} \approx 0.37
\]

**Exponential Graphs**

The exponential function \( f(x) = e^x \) is one of the most frequently occurring functions in calculus and its applications, so it is important to be familiar with its graph (Figure 14) and properties. We summarize these properties as follows, using the fact that this function is just a special case of the exponential functions considered in Theorem 2 but with base \( b = e > 1 \).
Chapter 6
Inverse Functions

10 Properties of the Natural Exponential Function
The exponential function $f(x) = e^x$ is an increasing continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. Thus $e^x > 0$ for all $x$. Also

$$
\lim_{x \to -\infty} e^x = 0 \quad \lim_{x \to \infty} e^x = \infty
$$

So the $x$-axis is a horizontal asymptote of $f(x) = e^x$.

**EXAMPLE 6** Find $\lim_{x \to \infty} \frac{e^{2x}}{e^{2x} + 1}$.

**SOLUTION** We divide numerator and denominator by $e^{2x}$:

$$
\lim_{x \to \infty} \frac{e^{2x}}{e^{2x} + 1} = \lim_{x \to \infty} \frac{1}{1 + e^{-2x}} = \frac{1}{1 + \lim_{x \to \infty} e^{-2x}}
$$

$$
= \frac{1}{1 + 0} = 1
$$

We have used the fact that $t = -2x \to -\infty$ as $x \to \infty$ and so

$$
\lim_{x \to \infty} e^{-2x} = \lim_{t \to -\infty} e^t = 0 \quad \blacksquare
$$

**EXAMPLE 7** Use the first and second derivatives of $f(x) = e^{1/x}$, together with asymptotes, to sketch its graph.

**SOLUTION** Notice that the domain of $f$ is $\{x \mid x \neq 0\}$, so we check for vertical asymptotes by computing the left and right limits as $x \to 0$. As $x \to 0^+$, we know that $t = 1/x \to \infty$, so

$$
\lim_{x \to 0^+} e^{1/x} = \lim_{t \to \infty} e^t = \infty
$$

and this shows that $x = 0$ is a vertical asymptote. As $x \to 0^-$, we have $t = 1/x \to -\infty$, so

$$
\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0
$$

As $x \to \pm \infty$, we have $1/x \to 0$ and so

$$
\lim_{x \to \pm \infty} e^{1/x} = e^0 = 1
$$

This shows that $y = 1$ is a horizontal asymptote (both to the left and right).

Now let’s compute the derivative. The Chain Rule gives

$$
f'(x) = -\frac{e^{1/x}}{x^2}
$$

Since $e^{1/x} > 0$ and $x^2 > 0$ for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$. Thus $f$ is decreasing on $(-\infty, 0)$ and on $(0, \infty)$. There is no critical number, so the function has no local maximum or minimum. The second derivative is

$$
f''(x) = -\frac{x^2 e^{1/x} (-1/x^2) - e^{1/x} (2x)}{x^4} = \frac{e^{1/x} (2x + 1)}{x^4}
$$
Since \(e^{3x} > 0\) and \(x^4 > 0\), we have \(f''(x) > 0\) when \(x > -\frac{1}{2}\) (\(x \neq 0\)) and \(f''(x) < 0\) when \(x < -\frac{1}{2}\). So the curve is concave downward on \((-\infty, -\frac{1}{2})\) and concave upward on \((-\frac{1}{2}, 0)\) and on \((0, \infty)\). The inflection point is \((-\frac{1}{2}, e^{-2})\).

To sketch the graph of \(f\) we first draw the horizontal asymptote \(y = 1\) (as a dashed line), together with the parts of the curve near the asymptotes in a preliminary sketch [Figure 15(a)]. These parts reflect the information concerning limits and the fact that \(f\) is decreasing on both \((-\infty, 0)\) and \((0, \infty)\). Notice that we have indicated that \(f(x) \to 0\) as \(x \to 0^\pm\) even though \(f(0)\) does not exist. In Figure 15(b) we finish the sketch by incorporating the information concerning concavity and the inflection point. In Figure 15(c) we check our work with a graphing device.

![Graphs showing the sketch of the function and its asymptotes.]

**FIGURE 15**

**Integration**

Because the exponential function \(y = e^x\) has a simple derivative, its integral is also simple:

\[
\int e^x \, dx = e^x + C
\]

**EXAMPLE 8** Evaluate \(\int x^2 e^{x^3} \, dx\).

**SOLUTION** We substitute \(u = x^3\). Then \(du = 3x^2 \, dx\), so \(x^2 \, dx = \frac{1}{3} \, du\) and

\[
\int x^2 e^{x^3} \, dx = \frac{1}{3} \int e^u \, du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C
\]

**EXAMPLE 9** Find the area under the curve \(y = e^{-3x}\) from 0 to 1.

**SOLUTION** The area is

\[
A = \int_0^1 e^{-3x} \, dx = \left[-\frac{1}{3}e^{-3x}\right]_0^1 = \frac{1}{3}(1 - e^{-3})
\]
6.2 EXERCISES

1. (a) Write an equation that defines the exponential function with base \( b > 0 \).
   (b) What is the domain of this function?
   (c) If \( b \neq 1 \), what is the range of this function?
   (d) Sketch the general shape of the graph of the exponential function for each of the following cases.
   (i) \( b > 1 \)
   (ii) \( b = 1 \)
   (iii) \( 0 < b < 1 \)

2. (a) How is the number \( e \) defined?
   (b) What is an approximate value for \( e \)?
   (c) What is the natural exponential function?

3–6 Graph the given functions on a common screen. How are these graphs related?
3. \( y = 2^x, \ y = e^x, \ y = 5^x, \ y = 20^x \)
4. \( y = e^x, \ y = e^{-x}, \ y = 8^x, \ y = 8^{-x} \)
5. \( y = 3^x, \ y = 10^x, \ y = (\frac{1}{2})^x, \ y = (\frac{1}{5})^x \)
6. \( y = 0.9^x, \ y = 0.6^x, \ y = 0.3^x, \ y = 0.1^x \)

7–12 Make a rough sketch of the graph of the function. Do not use a calculator. Just use the graphs given in Figures 3 and 14 and, if necessary, the transformations of Section 1.3.
7. \( y = 4^x - 1 \)
8. \( y = (0.5)^{x-1} \)
9. \( y = -2^{-x} \)
10. \( y = e^{x-1} \)
11. \( y = 1 - 2e^{-x} \)
12. \( y = 2(1 - e^x) \)

13. Starting with the graph of \( y = e^x \), write the equation of the graph that results from
   (a) shifting 2 units downward.
   (b) shifting 2 units to the right.
   (c) reflecting about the \( x \)-axis.
   (d) reflecting about the \( y \)-axis.
   (e) reflecting about the \( x \)-axis and then about the \( y \)-axis.

14. Starting with the graph of \( y = e^x \), find the equation of the graph that results from
   (a) reflecting about the line \( y = 4 \).
   (b) reflecting about the line \( x = 2 \).

15–16 Find the domain of each function.
15. (a) \( f(x) = \frac{1 - e^x}{1 - e^{-x}} \)
    (b) \( f(x) = \frac{1 + x}{e^{\cos x}} \)
16. (a) \( g(t) = \sqrt{10^t - 100} \)
    (b) \( g(t) = \sin(e^t - 1) \)

17–18 Find the exponential function \( f(x) = Cb^x \) whose graph is given.
17.
18.

19. Suppose the graphs of \( f(x) = x^2 \) and \( g(x) = 2^x \) are drawn on a coordinate grid where the unit of measurement is 1 inch. Show that, at a distance 2 ft to the right of the origin, the height of the graph of \( f \) is about 48 ft but the height of the graph of \( g \) is about 265 mi.

20. Compare the functions \( f(x) = x^2 \) and \( g(x) = 5^x \) by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place. Which function grows more rapidly when \( x \) is large?

21. Compare the functions \( f(x) = x^{10} \) and \( g(x) = e^x \) by graphing both \( f \) and \( g \) in several viewing rectangles. When does the graph of \( g \) finally surpass the graph of \( f \)?

22. Use a graph to estimate the values of \( x \) such that \( e^x > 1,000,000,000 \).

23–30 Find the limit.
23. \( \lim_{x \to \infty} (1.001)^x \)
24. \( \lim_{x \to \infty} (1.001)^x \)
25. \( \lim_{x \to \infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} \)
26. \( \lim_{x \to \infty} e^{x^2} \)
27. \( \lim_{x \to 2^+} e^{3/(2-x)} \)
28. \( \lim_{x \to 2^-} e^{3/(2-x)} \)
29. \( \lim_{x \to \infty} (e^{-2x} \cos x) \)
30. \( \lim_{x \to (\pi/2)^+} e^{\sin x} \)

31–50 Differentiate the function.
31. \( f(x) = e^5 \)
32. \( k(r) = e^r + r^e \)
33. \( f(x) = (3x^2 - 5x)e^x \)
34. \( y = \frac{e^x}{1 - e^x} \)
35. \( y = e^{ax^2} \)
36. \( g(x) = e^{ax} \)
37. \( y = e^{\sin \theta} \)
38. \( V(t) = \frac{4 + t}{te^t} \)
39. \( f(x) = \frac{x^2 e^x}{x^2 + e^x} \)
40. \( y = x^2 e^{-1/x} \)
41. \( y = x^2 e^{-3x} \)
42. \( f(t) = \tan(1 + e^{2t}) \)
43. \( f(t) = e^{at} \sin bt \)
44. \( f(z) = e^{z/(z-1)} \)
45. \( F(t) = e^{\sin 2t} \)  
46. \( y = e^{2t} + \sin(e^{2t}) \)  
47. \( g(u) = e^{\frac{1}{1+u^2}} \)  
48. \( y = \sqrt{1 + xe^{-2t}} \)  
49. \( y = \cos \left( \frac{1 - e^{2t}}{1 + e^{2t}} \right) \)  
50. \( f(t) = \sin^2(e^{2t^3}) \)

51–52. Find an equation of the tangent line to the curve at the given point.
51. \( y = e^{2t} \cos \pi x, \quad (0, 1) \)  
52. \( y = \frac{e^x}{x}, \quad (1, e) \)

53. Find \( y' \) if \( e^{x/y} = x - y \).
54. Find an equation of the tangent line to the curve \( xe^y + ye^x = 1 \) at the point \((0, 1)\).
55. Show that the function \( y = e^y + e^{-x/2} \) satisfies the differential equation \( 2y' - y' - y = 0 \).
56. Show that the function \( y = Ae^{x^2} + Bxe^{x^2} \) satisfies the differential equation \( y'' + 2y' + x^2y = 0 \).
57. For what values of \( r \) does the function \( y = e^{rx} \) satisfy the differential equation \( y'' + 6y' + 8y = 0 \)?
58. Find the values of \( \lambda \) for which \( y = e^{\lambda x} \) satisfies the equation \( y + y' = y'' \).
59. If \( f(x) = e^{2x} \), find a formula for \( f^{(n)}(x) \).
60. Find the thousandth derivative of \( f(x) = xe^{-x} \).

61. (a) Use the Intermediate Value Theorem to show that there is a root of the equation \( e^x + x = 0 \).  
(b) Use Newton’s method to find the root of the equation in part (a) correct to six decimal places.

62. Use a graph to find an initial approximation (to one decimal place) to the root of the equation \( 4e^{-x^2} \sin x = x^3 - x + 1 \). Then use Newton’s method to find the root correct to eight decimal places.

63. Use the graph of \( V \) in Figure 11 to estimate the half-life of the viral load of patient 303 during the first month of treatment.

64. A researcher is trying to determine the doubling time for a population of the bacterium \( Giardia \ lamblia \). He starts a culture in a nutrient solution and estimates the bacteria count every four hours. His data are shown in the table.

<table>
<thead>
<tr>
<th>Time (hours)</th>
<th>0</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bacteria CFU (CFU/mL)</td>
<td>37</td>
<td>47</td>
<td>63</td>
<td>78</td>
<td>105</td>
<td>130</td>
<td>173</td>
</tr>
</tbody>
</table>

(a) Make a scatter plot of the data.  
(b) Use a graphing calculator to find an exponential curve \( f(t) = a \cdot b^t \) that models the bacteria population \( t \) hours later.

65. Under certain circumstances a rumor spreads according to the equation \( p(t) = \frac{1}{1 + ae^{-kt}} \). 

\[ p(t) \] is the proportion of the population that has heard the rumor at time \( t \), and \( a \) and \( k \) are positive constants. [In Section 9.4 we will see that this is a reasonable model for \( p(t) \).]  
(a) Find \( \lim_{t \to \infty} p(t) \).  
(b) Find the rate of spread of the rumor.  
(c) Graph \( p \) for the case \( a = 10, k = 0.5 \) with \( t \) measured in hours. Use the graph to estimate how long it will take for 80% of the population to hear the rumor.

66. An object is attached to the end of a vibrating spring and its displacement from its equilibrium position is \( y = 8e^{-t^2/2} \sin 4t \), where \( t \) is measured in seconds and \( y \) is measured in centimeters.

(a) Graph the displacement function together with the functions \( y = 8e^{-t^2/2} \) and \( y = -8e^{-t^2/2} \). How are these graphs related? Can you explain why?  
(b) Use the graph to estimate the maximum value of the displacement. Does it occur when the graph touches the graph of \( y = 8e^{-t^2/2} \)?  
(c) What is the velocity of the object when it first returns to its equilibrium position?  
(d) Use the graph to estimate the time after which the displacement is no more than 2 cm from equilibrium.

67. Find the absolute maximum value of the function \( f(x) = x - e^x \).
68. Find the absolute minimum value of the function \( g(x) = e^x - x \).

69–70. Find the absolute maximum and absolute minimum values of \( f \) on the given interval.
69. \( f(x) = xe^{-x/2}, \quad [-1, 4] \)  
70. \( f(x) = xe^{x/2}, \quad [-3, 1] \)
71–72 Find (a) the intervals of increase or decrease, (b) the intervals of concavity, and (c) the points of inflection.

71. \( f(x) = (1 - x)e^{-x} \)  
72. \( f(x) = \frac{e^x}{x^2} \)

73–75 Discuss the curve using the guidelines of Section 3.5.

73. \( y = e^{-1/(x+1)} \)
74. \( y = e^{-x} \sin x, \quad 0 \leq x \leq 2\pi \)
75. \( y = 1/(1 + e^{-x}) \)

76. Let \( g(x) = e^{cx} + f(x) \) and \( h(x) = e^{kf(x)} \), where \( f(0) = 3 \), \( f'(0) = 5 \), and \( f''(0) = -2 \).
(a) Find \( g'(0) \) and \( g''(0) \) in terms of \( c \).
(b) In terms of \( k \), find an equation of the tangent line to the graph of \( h \) at the point where \( x = 0 \).

77. A drug response curve describes the level of medication in the bloodstream after a drug is administered. A surge function \( S(t) = At^pe^{-at} \) is often used to model the response curve, reflecting an initial surge in the drug level and then a more gradual decline. If, for a particular drug, \( A = 0.01 \), \( p = 4 \), \( k = 0.07 \), and \( t \) is measured in minutes, estimate the times corresponding to the inflection points and explain their significance. If you have a graphing device, use it to graph the drug response curve.

78. After an antibiotic tablet is taken, the concentration of the antibiotic in the bloodstream is modeled by the function

\[
C(t) = 8(e^{-0.4t} - e^{-0.6t})
\]

where the time \( t \) is measured in hours and \( C \) is measured in \( \mu g/mL \). What is the maximum concentration of the antibiotic during the first 12 hours?

79. After the consumption of an alcoholic beverage, the concentration of alcohol in the bloodstream (blood alcohol concentration, or BAC) surges as the alcohol is absorbed, followed by a gradual decline as the alcohol is metabolized. The function

\[
C(t) = 1.35te^{-2.802t}
\]

models the average BAC, measured in mg/mL, of a group of eight male subjects \( t \) hours after rapid consumption of 15 mL of ethanol (corresponding to one alcoholic drink). What is the maximum average BAC during the first 3 hours? When does it occur?


80–81 Draw a graph of \( f \) that shows all the important aspects of the curve. Estimate the local maximum and minimum values and then use calculus to find these values exactly. Use a graph of \( f'' \) to estimate the inflection points.

80. \( f(x) = e^{\cos x} \)  
81. \( f(x) = e^{3-x} \)

82. The family of bell-shaped curves

\[
y = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}
\]

occurs in probability and statistics, where it is called the normal density function. The constant \( \mu \) is called the mean and the positive constant \( \sigma \) is called the standard deviation. For simplicity, let’s scale the function so as to remove the factor \( 1/(\sqrt{2\pi}) \) and let’s analyze the special case where \( \mu = 0 \). So we study the function

\[
f(x) = e^{-x^2/(2\sigma^2)}
\]

(a) Find the asymptote, maximum value, and inflection points of \( f \).
(b) What role does \( \sigma \) play in the shape of the curve?
(c) Illustrate by graphing four members of this family on the same screen.

83–94 Evaluate the integral.

83. \( \int_0^1 (x^e + e^x) \, dx \)
84. \( \int_0^e e^x \, dx \)
85. \( \int_0^2 \frac{dx}{e^{x^2}} \)
86. \( \int x^2 e^{x^3} \, dx \)
87. \( \int e^x \sqrt{1 + e^x} \, dx \)
88. \( \int \frac{(1 + e^x)^2}{e^x} \, dx \)
89. \( \int (e^x + e^{-x})^2 \, dx \)
90. \( \int e^x(4 + e^{-x})^3 \, dx \)
91. \( \int \frac{e^x}{1 - e^{2x}} \, du \)
92. \( \int e^x \sin \theta \cos \theta \, d\theta \)
93. \( \int_1^2 \frac{e^{1/x}}{x^2} \, dx \)
94. \( \int_0^1 \frac{\sqrt{1 + e^{-x}}}{e^x} \, dx \)

95. Find, correct to three decimal places, the area of the region bounded by the curves \( y = e^x \), \( y = e^{-x} \), and \( x = 1 \).
96. Find \( f(x) \) if \( f''(x) = 3e^x + 5 \sin x \), \( f(0) = 1 \), and \( f'(0) = 2 \).
97. Find the volume of the solid obtained by rotating about the \( x \)-axis the region bounded by the curves \( y = e^x \), \( y = 0 \), \( x = 0 \), and \( x = 1 \).
98. Find the volume of the solid obtained by rotating about the \( y \)-axis the region bounded by the curves \( y = e^{-x^2} \), \( y = 0 \), \( x = 0 \), and \( x = 1 \).
99. The error function

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt
\]
100. Show that the function

\[ y = e^x \text{erf}(x) \]

satisfies the differential equation

\[ y' = 2xy + 2/\sqrt{\pi} \]

101. An oil storage tank ruptures at time \( t = 0 \) and oil leaks from the tank at a rate of \( r(t) = 100e^{-0.0t} \) liters per minute. How much oil leaks out during the first hour?

102. A bacteria population starts with 400 bacteria and grows at a rate of \( r(t) = (450.268)e^{1.12567t} \) bacteria per hour. How many bacteria will there be after three hours?

103. Dialysis treatment removes urea and other waste products from a patient’s blood by diverting some of the bloodflow externally through a machine called a dialyzer. The rate at which urea is removed from the blood (in mg/min) is often well described by the equation

\[ u(t) = \frac{r}{V} C_0 e^{-t/V} \]

where \( r \) is the rate of flow of blood through the dialyzer (in mL/min), \( V \) is the volume of the patient’s blood (in mL), and \( C_0 \) is the amount of urea in the blood (in mg) at time \( t = 0 \). Evaluate the integral \( \int_0^3 u(t) \, dt \) and interpret it.

104. The rate of growth of a fish population was modeled by the equation

\[ G(t) = \frac{60,000e^{-0.06t}}{(1 + 5e^{-0.05t})^2} \]

where \( t \) is measured in years and \( G \) in kilograms per year. If the biomass (the total mass of the population) was 25,000 kg in the year 2000, what is the predicted biomass for the year 2020?

105. If \( f(x) = 3 + x + e^x \), find \( (f^{-1})'(4) \).

106. Evaluate \( \lim_{x \to \pi} \frac{e^{\sin x} - 1}{x - \pi} \).

107. If you graph the function

\[ f(x) = \frac{1 - e^{1/x}}{1 + e^{1/x}} \]

you’ll see that \( f \) appears to be an odd function. Prove it.

108. Graph several members of the family of functions

\[ f(x) = \frac{1}{1 + ae^{bx}} \]

where \( a > 0 \). How does the graph change when \( b \) changes? How does it change when \( a \) changes?

109. (a) Show that \( e^x \geq 1 + x \) if \( x \geq 0 \).

[Hint: Show that \( f(x) = e^x - (1 + x) \) is increasing for \( x > 0 \).]

(b) Deduce that \( 1/2 \leq \int_0^1 e^x \, dx \leq e \).

110. (a) Use the inequality of Exercise 109(a) to show that, for \( x \geq 0 \),

\[ e^x \geq 1 + x + \frac{1}{2}x^2 \]

(b) Use part (a) to improve the estimate of \( \int_0^1 e^x \, dx \) given in Exercise 109(b).

111. (a) Use mathematical induction to prove that for \( x \geq 0 \) and any positive integer \( n \),

\[ e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \]

(b) Use part (a) to show that \( e > 2.7 \).

(c) Use part (a) to show that

\[ \lim_{x \to \infty} \frac{e^x}{x^k} = \infty \]

for any positive integer \( k \).

---

**6.3 Logarithmic Functions**

If \( b > 0 \) and \( b \neq 1 \), the exponential function \( f(x) = b^x \) is either increasing or decreasing and so it is one-to-one by the Horizontal Line Test. It therefore has an inverse function \( f^{-1} \), which is called the **logarithmic function with base** \( b \) and is denoted by \( \log_b \). If we use the formulation of an inverse function given by (6.1.3),

\[ f^{-1}(x) = y \iff f(y) = x \]

then we have

\[ \log_b x = y \iff b^y = x \]

Thus, if \( x > 0 \), then \( \log_b x \) is the exponent to which the base \( b \) must be raised to give \( x \).
EXAMPLE 1  Evaluate (a) \( \log_3 81 \), (b) \( \log_{25} 5 \), and (c) \( \log_{10} 0.001 \).

SOLUTION
(a) \( \log_3 81 \)  because \( 3^4 = 81 \)
(b) \( \log_{25} 5 \)  because \( 25^{1/2} = 5 \)
(c) \( \log_{10} 0.001 \)  because \( 10^{-3} = 0.001 \)

The cancellation equations (6.1.4), when applied to the functions \( f(x) = b^x \) and \( f^{-1}(x) = \log_b x \), become

\[
\log_b b^x = x \quad \text{for every } x \in \mathbb{R}
\]
\[
b^{\log_b x} = x \quad \text{for every } x > 0
\]

The logarithmic function \( \log_b \) has domain \((0, \infty)\) and range \(\mathbb{R} \) and is continuous since it is the inverse of a continuous function, namely, the exponential function. Its graph is the reflection of the graph of \( y = b^x \) about the line \( y = x \).

Figure 1 shows the case where \( b > 1 \). (The most important logarithmic functions have base \( b > 1 \).) The fact that \( y = b^x \) is a very rapidly increasing function for \( x > 0 \) is reflected in the fact that \( y = \log_b x \) is a very slowly increasing function for \( x > 1 \).

Figure 2 shows the graphs of \( y = \log_b x \) with various values of the base \( b > 1 \). Since \( \log_b 1 = 0 \), the graphs of all logarithmic functions pass through the point \((1, 0)\).

The following theorem summarizes the properties of logarithmic functions.

**Theorem**  If \( b > 1 \), the function \( f(x) = \log_b x \) is a one-to-one, continuous, increasing function with domain \((0, \infty)\) and range \(\mathbb{R} \). If \( x, y > 0 \) and \( r \) is any real number, then

1. \( \log_b(xy) = \log_b x + \log_b y \)
2. \( \log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y \)
3. \( \log_b(x^r) = r \log_b x \)

Properties 1, 2, and 3 follow from the corresponding properties of exponential functions given in Theorem 6.2.2.
EXAMPLE 2 Use the properties of logarithms in Theorem 3 to evaluate the following.
(a) $\log_2 2 + \log_4 32$  (b) $\log_2 80 - \log_2 5$

SOLUTION  
(a) Using Property 1 in Theorem 3, we have  
$$\log_2 2 + \log_4 32 = \log_4 (2 \cdot 32) = \log_4 64 = 3$$  
since $4^3 = 64$.

(b) Using Property 2 we have  
$$\log_2 80 - \log_2 5 = \log_2 \left( \frac{80}{5} \right) = \log_2 16 = 4$$  
since $2^4 = 16$.

The limits of exponential functions given in Section 6.2 are reflected in the following limits of logarithmic functions. (Compare with Figure 1.)

4 If $b > 1$, then  
$$\lim_{x \to \infty} \log_b x = \infty \quad \text{and} \quad \lim_{x \to 0^+} \log_b x = -\infty$$

In particular, the $y$-axis is a vertical asymptote of the curve $y = \log_b x$.

EXAMPLE 3 Find $\lim_{x \to 0} \log_{10} (\tan^2 x)$.

SOLUTION As $x \to 0$, we know that $t = \tan^2 x \to \tan^2 0 = 0$ and the values of $t$ are positive. So by (4) with $b = 10 > 1$, we have  
$$\lim_{x \to 0} \log_{10} (\tan^2 x) = \lim_{t \to 0^+} \log_{10} t = -\infty$$

Natural Logarithms

Of all possible bases $b$ for logarithms, we will see in the next section that the most convenient choice of a base is the number $e$, which was defined in Section 6.2. The logarithm with base $e$ is called the natural logarithm and has a special notation:

$$\log_e x = \ln x$$

If we put $b = e$ and replace log, with “ln” in (1) and (2), then the defining properties of the natural logarithm function become

5  
$$\ln x = y \iff e^y = x$$

6  
$$\ln (e^x) = x \quad x \in \mathbb{R}$$  
$$e^{\ln x} = x \quad x > 0$$

In particular, if we set $x = 1$, we get  
$$\ln e = 1$$
EXAMPLE 4 Find \( x \) if \( \ln x = 5 \).

SOLUTION 1 From (5) we see that
\[
\ln x = 5 \quad \text{means} \quad e^5 = x
\]
Therefore \( x = e^5 \).

(If you have trouble working with the “\( \ln \)” notation, just replace it by \( \log_e \). Then the equation becomes \( \log_e x = 5 \); so, by the definition of logarithm, \( e^5 = x \).)

SOLUTION 2 Start with the equation
\[
\ln x = 5
\]
and apply the exponential function to both sides of the equation:
\[
e^{\ln x} = e^5
\]
But the second cancellation equation in (6) says that \( e^{\ln x} = x \). Therefore \( x = e^5 \). —

EXAMPLE 5 Solve the equation \( e^{5-3x} = 10 \).

SOLUTION We take natural logarithms of both sides of the equation and use (6):
\[
\ln(e^{5-3x}) = \ln 10
\]
\[
5 - 3x = \ln 10
\]
\[
3x = 5 - \ln 10
\]
\[
x = \frac{1}{3}(5 - \ln 10)
\]
Since the natural logarithm is found on scientific calculators, we can approximate the solution: to four decimal places, \( x \approx 0.8991 \). —

EXAMPLE 6 Express \( \ln a + \frac{1}{2} \ln b \) as a single logarithm.

SOLUTION Using Properties 3 and 1 of logarithms, we have
\[
\ln a + \frac{1}{2} \ln b = \ln a + \ln b^{1/2}
\]
\[
= \ln a + \ln \sqrt{b}
\]
\[
= \ln(a \sqrt{b})
\]

The following formula shows that logarithms with any base can be expressed in terms of the natural logarithm.

7 Change of Base Formula For any positive number \( b \) \((b \neq 1)\), we have
\[
\log_b x = \frac{\ln x}{\ln b}
\]

PROOF Let \( y = \log_b x \). Then, from (1), we have \( b^y = x \). Taking natural logarithms of both sides of this equation, we get \( y \ln b = \ln x \). Therefore
\[
y = \frac{\ln x}{\ln b}
\]
Scientific calculators have a key for natural logarithms, so Formula 7 enables us to use a calculator to compute a logarithm with any base (as shown in the following example). Similarly, Formula 7 allows us to graph any logarithmic function on a graphing calculator or computer (see Exercises 20–22).

**EXAMPLE 7** Evaluate \( \log_8 5 \) correct to six decimal places.

**SOLUTION** Formula 7 gives

\[
\log_8 5 = \frac{\ln 5}{\ln 8} 
\]

\[\approx 0.773976\]

**Graph and Growth of the Natural Logarithm**

The graphs of the exponential function \( y = e^x \) and its inverse function, the natural logarithm function, are shown in Figure 3. Because the curve \( y = e^x \) crosses the \( y \)-axis with a slope of 1, it follows that the reflected curve \( y = \ln x \) crosses the \( x \)-axis with a slope of 1.

In common with all other logarithmic functions with base greater than 1, the natural logarithm is a continuous, increasing function defined on \( (0, \infty) \) and the \( y \)-axis is a vertical asymptote.

If we put \( b = e \) in (4), then we have the following limits:

\[
\lim_{x \to \infty} \ln x = \infty \quad \lim_{x \to 0^+} \ln x = -\infty
\]

**EXAMPLE 8** Sketch the graph of the function \( y = \ln(x - 2) - 1 \).

**SOLUTION** We start with the graph of \( y = \ln x \) as given in Figure 3. Using the transformations of Section 1.3, we shift it 2 units to the right to get the graph of \( y = \ln(x - 2) \) and then we shift it 1 unit downward to get the graph of \( y = \ln(x - 2) - 1 \). (See Figure 4.)

Notice that the line \( x = 2 \) is a vertical asymptote since

\[
\lim_{x \to 2^+} [\ln(x - 2) - 1] = -\infty
\]

We have seen that \( \ln x \to \infty \) as \( x \to \infty \). But this happens very slowly. In fact, \( \ln x \) grows more slowly than any positive power of \( x \). To illustrate this fact, we compare approximate values of the functions \( y = \ln x \) and \( y = x^{1/2} = \sqrt{x} \) in the following table and we graph them in Figures 5 and 6 on page 426. You can see that initially the graphs of \( y = \sqrt{x} \) and \( y = \ln x \) grow at comparable rates, but eventually the root function far surpasses the logarithm. In fact, we will be able to show in Section 6.8 that

\[
\lim_{x \to \infty} \frac{\ln x}{x^p} = 0
\]
for any positive power \( p \). So for large \( x \), the values of \( \ln x \) are very small compared with \( x^p \). (See Exercise 72.)

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
x & 1 & 2 & 5 & 10 & 50 & 100 & 500 & 1000 & 10,000 & 100,000 \\
\hline
\ln x & 0 & 0.69 & 1.61 & 2.30 & 3.91 & 4.6 & 6.2 & 6.9 & 9.2 & 11.5 \\
\hline
\frac{\sqrt{x}}{x} & 1 & 1.41 & 2.24 & 3.16 & 7.07 & 10.0 & 22.4 & 31.6 & 100 & 316 \\
\hline
\end{array}
\]

\( \text{FIGURE 5} \)

\( \text{FIGURE 6} \)

### 6.3 Exercises

1. (a) How is the logarithmic function \( y = \log_b x \) defined?
   (b) What is the domain of this function?
   (c) What is the range of this function?
   (d) Sketch the general shape of the graph of the function \( y = \log_b x \) if \( b > 1 \).

2. (a) What is the natural logarithm?
   (b) What is the common logarithm?
   (c) Sketch the graphs of the natural logarithm function and the natural exponential function with a common set of axes.

3–8 Find the exact value of each expression.

3. (a) \( \log_2 32 \)  
   (b) \( \log_8 2 \)

4. (a) \( \log_5 \frac{1}{125} \)  
   (b) \( \ln(1/e^2) \)

5. (a) \( e^{\ln 4.5} \)  
   (b) \( \log_{10} 0.0001 \)

6. (a) \( \log_{1.5} 2.25 \)  
   (b) \( \log_5 4 - \log_5 500 \)

7. (a) \( \log_{10} 40 + \log_{10} 2.5 \)  
   (b) \( \log_8 60 - \log_8 3 - \log_8 5 \)

8. (a) \( e^{-\ln 2} \)  
   (b) \( e^{\ln(e^3)} \)

9–12 Use the properties of logarithms to expand the quantity.

9. \( \ln \sqrt{ab} \)  
10. \( \log_{10} \sqrt{\frac{x-1}{x+1}} \)

11. \( \ln \frac{x^2}{y^{2/3}} \)  
12. \( \ln(s^4 \sqrt{t/\sqrt{u}}) \)

13–18 Express the quantity as a single logarithm.

13. \( 2 \ln x + 3 \ln y - \ln z \)
14. \( \log_{10} 4 + \log_{10} a - \frac{1}{2} \log_{10}(a + 1) \)
15. \( \ln 10 + 2 \ln 5 \)  
16. \( \ln 3 + \frac{1}{2} \ln 8 \)
17. \( \frac{1}{3} \ln(x + 2)^3 + \frac{1}{2} \ln(x - \ln(x^2 + 3x + 2^2)) \)
18. \( \ln b + 2 \ln c - 3 \ln d \)

19. Use Formula 7 to evaluate each logarithm correct to six decimal places.
   (a) \( \log_3 10 \)  
   (b) \( \log_3 57 \)  
   (c) \( \log_2 \pi \)

20–22 Use Formula 7 to graph the given functions on a common screen. How are these graphs related?

20. \( y = \log_{10} x \), \( y = \log_8 x \), \( y = \log_{1.5} x \), \( y = \log_{10} x \), \( y = \log_{\log_{10}x} x \), \( y = \log_{10} x \), \( y = \log_{10} x \), \( y = e^x \), \( y = 10^x \)
23–24 Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graphs given in Figures 2 and 3 and, if necessary, the transformations of Section 1.3.

23. (a) \( y = \log_{10}(x + 5) \)  
   (b) \( y = -\ln x \)  
24. (a) \( y = \ln(-x) \)  
   (b) \( y = \ln |x| \)

25–26
(a) What are the domain and range of \( f \)?
(b) What is the \( x \)-intercept of the graph of \( f \)?
(c) Sketch the graph of \( f \).
25. \( f(x) = \ln x + 2 \)  
26. \( f(x) = \ln(x - 1) - 1 \)

27–36 Solve each equation for \( x \).
27. (a) \( e^{x-2} = 6 \)  
   (b) \( \ln(3x - 10) = 2 \)
28. (a) \( \ln(x^2 - 1) = 3 \)  
   (b) \( e^{2x} - 3e^x + 2 = 0 \)
29. (a) \( 2^{x^2} = 3 \)  
   (b) \( \ln x + \ln(x - 1) = 1 \)
30. (a) \( e^{3x+1} = k \)  
   (b) \( \log_5(mx) = c \)
31. \( e^x - e^{-2x} = 1 \)  
32. \( 10(1 + e^{-x})^{-1} = 3 \)
33. \( \ln(\ln x) = 1 \)  
34. \( e^{3x} = 10 \)
35. \( e^{2x} - e^x - 6 = 0 \)  
36. \( \ln(2x + 1) = 2 - \ln x \)

37–38 Find the solution of the equation correct to four decimal places.
37. (a) \( \ln(1 + x^3) - 4 = 0 \)  
   (b) \( 2e^{1/2} = 42 \)
38. (a) \( 2^{x^2 - 3x} = 99 \)  
   (b) \( \ln\left(\frac{x + 1}{x}\right) = 2 \)

39–40 Solve each inequality for \( x \).
39. (a) \( \ln x < 0 \)  
   (b) \( e^x > 5 \)
40. (a) \( 1 < e^{3x-1} < 2 \)  
   (b) \( 1 - 2 \ln x < 3 \)

41. Suppose that the graph of \( y = \log_2 x \) is drawn on a coordinate grid where the unit of measurement is an inch. How many miles to the right of the origin do we have to move before the height of the curve reaches 3 ft?

42. The velocity of a particle that moves in a straight line under the influence of viscous forces is \( v(t) = e^{-kt} \), where \( c \) and \( k \) are positive constants.
   (a) Show that the acceleration is proportional to the velocity.
   (b) Explain the significance of the number \( c \).
   (c) At what time is the velocity equal to half the initial velocity?

43. The geologist C. F. Richter defined the magnitude of an earthquake to be \( \log_{10}(I/S) \), where \( I \) is the intensity of the
   earthquake (measured by the amplitude of a seismograph 100 km from the epicenter) and \( S \) is the intensity of a "standard" earthquake (where the amplitude is only 1 micron = \( 10^{-4} \) cm).
   The 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. The 1906 San Francisco earthquake was 16 times as intense. What was its magnitude on the Richter scale?

44. A sound so faint that it can just be heard has intensity \( I_0 = 10^{-12} \) watt/m² at a frequency of 1000 hertz (Hz). The loudness, in decibels (dB), of a sound with intensity \( I \) is then defined to be \( L = 10 \log_{10}(I/I_0) \). Amplified rock music is measured at 120 dB, whereas the noise from a motor-driven lawn mower is measured at 106 dB. Find the ratio of the intensity of the rock music to that of the mower.

45. If a bacteria population starts with 100 bacteria and doubles every three hours, then the number of bacteria after \( t \) hours is \( n = f(t) = 100 \cdot 2^{t/3} \).
   (a) Find the inverse of this function and explain its meaning.
   (b) When will the population reach 50,000?

46. When a camera flash goes off, the batteries immediately begin to recharge the flash’s capacitor, which stores electric charge given by
   \[ Q(t) = Q_0(1 - e^{-lt/c}) \]
   (The maximum charge capacity is \( Q_0 \) and \( t \) is measured in seconds.)
   (a) Find the inverse of this function and explain its meaning.
   (b) How long does it take to recharge the capacitor to 90% of its capacity if \( a = 2 \)?

47–52 Find the limit.
47. \( \lim_{x \to 3} \ln(x^2 - 9) \)  
48. \( \lim_{x \to 0} \log_3(8x - x^4) \)
49. \( \lim_{x \to 0} \ln(\cos x) \)  
50. \( \lim_{x \to 0^+} \ln(\sin x) \)
51. \( \lim_{x \to \infty} [\ln(1 + x^2) - \ln(1 + x)] \)
52. \( \lim_{x \to \infty} [\ln(2 + x) - \ln(1 + x)] \)

53–54 Find the domain of the function.
53. \( f(x) = \ln(4 - x^2) \)  
54. \( g(x) = \log_2(x^2 + 3x) \)

55–57 Find (a) the domain of \( f \) and (b) \( f^{-1} \) and its domain.
55. \( f(x) = \sqrt{3 - e^{2x}} \)  
56. \( f(x) = \ln(2 + \ln x) \)
57. \( f(x) = \ln(e^x - 3) \)

58. (a) What are the values of \( e^{\ln(300)} \) and \( \ln(e^{300}) \)?
   (b) Use your calculator to evaluate \( e^{\ln(300)} \) and \( \ln(e^{300}) \). What do you notice? Can you explain why the calculator has trouble?
59–64 Find the inverse function.

59. \( y = 2 \ln(x - 1) \)  
60. \( g(x) = \log_2(x^3 + 2) \)  
61. \( f(x) = e^{x^2} \)  
62. \( y = (\ln x)^3, \ x \geq 1 \)  
63. \( y = 3^2x - 4 \)  
64. \( y = \frac{1 - e^{-x}}{1 + e^{-x}} \)

65. On what interval is the function \( f(x) = e^{2x} - e^x \) increasing?

66. On what interval is the curve \( y = 2e^x - e^{-3x} \) concave downward?

67. (a) Show that the function \( f(x) = \ln(x + \sqrt{x^2 + 1}) \) is an odd function.
(b) Find the inverse function of \( f \).

68. Find an equation of the tangent to the curve \( y = e^{-x} - x \) that is perpendicular to the line \( 2x - y = 8 \).

69. Show that the equation \( x^{1/\ln x} = 2 \) has no solution. What can you say about the function \( f(x) = x^{1/\ln x} \)?

70. Any function of the form \( f(x) = [g(x)]^{\ln x} \), where \( g(x) > 0 \), can be analyzed as a power of \( e \) by writing \( g(x) = e^{\ln g(x)} \) so that \( f(x) = e^{\ln x \ln g(x)} \). Using this device, calculate each limit.
(a) \( \lim_{x \to -\infty} x^{\ln x} \)
(b) \( \lim_{x \to 0^+} x^{-\ln x} \)
(c) \( \lim_{x \to 0^+} x^{1/x} \)
(d) \( \lim_{x \to -\infty} (\ln 2)x^{-\ln x} \)

71. Let \( b > 1 \). Prove, using Definitions 3.4.6 and 3.4.7, that
(a) \( \lim_{x \to -\infty} b^x = 0 \)
(b) \( \lim_{x \to -\infty} b^x \to \infty \)
(c) Any function of the form \( f(x) = x^{0.1} \) and \( g(x) = \ln x \) by graphing both \( f \) and \( g \) in several viewing rectangles. When does the graph of \( f \) finally surpass the graph of \( g \)?
(b) Graph the function \( h(x) = (\ln x)/x^{0.1} \) in a viewing rectangle that displays the behavior of the function as \( x \to \infty \).

72. (a) Compare the rates of growth of \( f(x) = x^{0.1} \) and \( g(x) = \ln x \) by graphing both \( f \) and \( g \) in several viewing rectangles. When does the graph of \( f \) finally surpass the graph of \( g \)?

(c) Find a number \( N \) such that
\[
\frac{\ln x}{x^{0.1}} < 0.1
\]
if \( x > N \)

73. Solve the inequality \( \ln(x^2 - 2x - 2) \leq 0 \).

74. A prime number is a positive integer that has no factors other than 1 and itself. The first few primes are 2, 3, 5, 7, 11, 13, 17, . . . . We denote by \( \pi(n) \) the number of primes that are less than or equal to \( n \). For instance, \( \pi(15) = 6 \) because there are six primes smaller than 15.

(a) Calculate the numbers \( \pi(25) \) and \( \pi(100) \).

(b) By inspecting tables of prime numbers and tables of logarithms, the great mathematician K. F. Gauss made the guess in 1792 (when he was 15) that the number of primes up to \( n \) is approximately \( n/\ln n \) when \( n \) is large. More precisely, he conjectured that
\[
\lim_{n \to \infty} \frac{\pi(n)}{n/\ln n} = 1
\]
This was finally proved, a hundred years later, by Jacques Hadamard and Charles de la Vallée Poussin and is called the Prime Number Theorem. Provide evidence for the truth of this theorem by computing the ratio of \( \pi(n) \) to \( n/\ln n \) for \( n = 100, 1000, 10^4, 10^5, 10^6, \) and \( 10^7 \). Use the following data: \( \pi(1000) = 168, \pi(10^4) = 1229, \pi(10^5) = 9592, \pi(10^6) = 78,498, \pi(10^7) = 664,579 \).

(c) Use the Prime Number Theorem to estimate the number of primes up to a billion.

6.4 Derivatives of Logarithmic Functions

In this section we find the derivatives of the logarithmic functions \( y = \log_b x \) and the exponential functions \( y = b^x \). We start with the natural logarithmic function \( y = \ln x \).

We know that it is differentiable because it is the inverse of the differentiable function \( y = e^x \).

PROOF Let \( y = \ln x \). Then
\[
\frac{d}{dx} (\ln x) = \frac{1}{x}
\]

\[
e^y = x
\]
Differentiating this equation implicitly with respect to \( x \), we get
\[
e^y \frac{dy}{dx} = 1
\]
and so
\[
\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}
\]

**EXAMPLE 1** Differentiate \( y = \ln(x^3 + 1) \).

**SOLUTION** To use the Chain Rule, we let \( u = x^3 + 1 \). Then \( y = \ln u \), so
\[
\frac{dy}{dx} = \frac{1}{u} \frac{du}{dx} = \frac{1}{x^3 + 1} (3x^2) = \frac{3x^2}{x^3 + 1}
\]

In general, if we combine Formula 1 with the Chain Rule as in Example 1, we get
\[
\frac{d}{dx} \ln(u) = \frac{1}{u} \frac{du}{dx} \quad \text{or} \quad \frac{d}{dx} \ln(g(x)) = \frac{g'(x)}{g(x)}
\]

**EXAMPLE 2** Find \( \frac{d}{dx} \ln(\sin x) \).

**SOLUTION** Using (2), we have
\[
\frac{d}{dx} \ln(\sin x) = \frac{1}{\sin x} \frac{d}{dx} (\sin x) = \frac{1}{\sin x} \cos x = \cot x
\]

**EXAMPLE 3** Differentiate \( f(x) = \sqrt{\ln x} \).

**SOLUTION** This time the logarithm is the inner function, so the Chain Rule gives
\[
f''(x) = \frac{1}{2}(\ln x)^{-1/2} \frac{d}{dx} (\ln x) = \frac{1}{2 \sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x \sqrt{\ln x}}
\]

**EXAMPLE 4** Find \( \frac{d}{dx} \ln \frac{x + 1}{\sqrt{x - 2}} \).

**SOLUTION**
\[
\frac{d}{dx} \ln \frac{x + 1}{\sqrt{x - 2}} = \frac{1}{x + 1} \frac{d}{dx} \frac{x + 1}{\sqrt{x - 2}}
\]
\[
= \frac{\sqrt{x - 2}}{x + 1} \cdot \frac{x - (x + 1)(\frac{1}{2})(x - 2)^{-1/2}}{x - 2}
\]
\[
= \frac{x - 2 - \frac{1}{2}(x + 1)(x - 2)^{-1/2}}{(x + 1)(x - 2)}
\]
\[
= \frac{2x - 5}{2(x + 1)(x - 2)}
\]
**Solution 2** If we first simplify the given function using the properties of logarithms, then the differentiation becomes easier:

\[
\frac{d}{dx} \ln \frac{x + 1}{\sqrt{x + 2}} = \frac{d}{dx} \left[ \ln(x + 1) - \frac{1}{2} \ln(x - 2) \right]
\]

\[
= \frac{1}{x + 1} - \frac{1}{2} \left( \frac{1}{x - 2} \right)
\]

(This answer can be left as written, but if we used a common denominator we would see that it gives the same answer as in Solution 1.)

**Example 5** Find the absolute minimum value of \( f(x) = x^2 \ln x \).

**Solution** The domain is \((0, \infty)\) and the Product Rule gives

\[
f'(x) = x^2 \cdot \frac{1}{x} + 2x \ln x = x(1 + 2 \ln x)
\]

Therefore \( f''(x) = 0 \) when \( 2 \ln x = -1 \), that is, \( \ln x = -\frac{1}{2} \), or \( x = e^{-1/2} \). Also, \( f''(x) > 0 \) when \( x > e^{-1/2} \) and \( f''(x) < 0 \) for \( 0 < x < e^{-1/2} \). So, by the First Derivative Test for Absolute Extreme Values, \( f(1/\sqrt{e}) = -1/(2e) \) is the absolute minimum.

**Example 6** Discuss the curve \( y = \ln(4 - x^2) \) using the guidelines of Section 3.5.

A. The domain is

\[
\{ x \mid 4 - x^2 > 0 \} = \{ x \mid x^2 < 4 \} = \{ x \mid |x| < 2 \} = (-2, 2)
\]

B. The y-intercept is \( f(0) = \ln 4 \). To find the x-intercept we set

\[
y = \ln(4 - x^2) = 0
\]

We know that \( \ln 1 = \log_e 1 = 0 \) (since \( e^0 = 1 \)), so we have \( 4 - x^2 = 1 \) \( \Rightarrow \) \( x^2 = 3 \) and therefore the x-intercepts are \( \pm \sqrt{3} \).

C. Since \( f(-x) = f(x) \), \( f \) is even and the curve is symmetric about the y-axis.

D. We look for vertical asymptotes at the endpoints of the domain. Since \( 4 - x^2 \to 0^+ \) as \( x \to 2^- \) and also as \( x \to -2^+ \), we have

\[
\lim_{x \to 2^-} \ln(4 - x^2) = -\infty \quad \lim_{x \to -2^+} \ln(4 - x^2) = -\infty
\]

by (6.3.8). Thus the lines \( x = 2 \) and \( x = -2 \) are vertical asymptotes.

E.

\[
f''(x) = \frac{-2x}{4 - x^2}
\]

Since \( f''(x) > 0 \) when \(-2 < x < 0\) and \( f''(x) < 0 \) when \( 0 < x < 2 \), \( f \) is increasing on \((-2, 0)\) and decreasing on \((0, 2)\).

F. The only critical number is \( x = 0 \). Since \( f' \) changes from positive to negative at 0, \( f(0) = \ln 4 \) is a local maximum by the First Derivative Test.

G.

\[
f'''(x) = \frac{(4 - x^2)(-2) + 2x(-2x)}{(4 - x^2)^2} = \frac{-8 - 2x^2}{(4 - x^2)^2}
\]
Since \( f''(x) < 0 \) for all \( x \), the curve is concave downward on \((-2, 2)\) and has no inflection point.

**H.** Using this information, we sketch the curve in Figure 2.

**EXAMPLE 7** Find \( f'(x) \) if \( f(x) = \ln|x| \).

**SOLUTION** Since

\[
f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}
\]

it follows that

\[
f'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{-x}(-1) = \frac{1}{x} & \text{if } x < 0 \end{cases}
\]

Thus \( f'(x) = \frac{1}{x} \) for all \( x \neq 0 \).

The result of Example 7 is worth remembering:

\[
\frac{d}{dx}(\ln|x|) = \frac{1}{x}
\]

The corresponding integration formula is

\[
\int \frac{1}{x} \, dx = \ln|x| + C
\]

Notice that this fills the gap in the rule for integrating power functions:

\[
\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad \text{if } n \neq -1
\]

The missing case \((n = -1)\) is supplied by Formula 4.

**EXAMPLE 8** Find, correct to three decimal places, the area of the region under the hyperbola \( xy = 1 \) from \( x = 1 \) to \( x = 2 \).
SOLUTION The given region is shown in Figure 4. Using Formula 4 (without the absolute value sign, since $x > 0$), we see that the area is
\[
A = \int_1^2 \frac{1}{x} \, dx = \ln x \bigg|_1^2 = \ln 2 - \ln 1 = \ln 2 \approx 0.693
\]

EXAMPLE 9 Evaluate \( \int \frac{x}{x^2 + 1} \, dx \).

SOLUTION We make the substitution \( u = x^2 + 1 \) because the differential \( du = 2x \, dx \) occurs (except for the constant factor 2). Thus \( x \, dx = \frac{1}{2} \, du \) and
\[
\int \frac{x}{x^2 + 1} \, dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C
\]
\[
= \frac{1}{2} \ln |x^2 + 1| + C = \frac{1}{2} \ln(x^2 + 1) + C
\]
Notice that we removed the absolute value signs because \( x^2 + 1 > 0 \) for all \( x \). We could use the properties of logarithms to write the answer as
\[
\ln \sqrt{x^2 + 1} + C
\]
but this isn’t necessary.

EXAMPLE 10 Calculate \( \int_1^e \frac{\ln x}{x} \, dx \).

SOLUTION We let \( u = \ln x \) because its differential \( du = dx/x \) occurs in the integral. When \( x = 1, u = \ln 1 = 0 \); when \( x = e, u = \ln e = 1 \). Thus
\[
\int_1^e \frac{\ln x}{x} \, dx = \int_0^1 u \, du = \frac{u^2}{2} \bigg|_0^1 = \frac{1}{2}
\]

EXAMPLE 11 Calculate \( \int \tan x \, dx \).

SOLUTION First we write tangent in terms of sine and cosine:
\[
\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx
\]
This suggests that we should substitute \( u = \cos x \), since then \( du = -\sin x \, dx \) and so \( \sin x \, dx = -\, du \):
\[
\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{1}{u} \, du
\]
\[
= -\ln |u| + C = -\ln |\cos x| + C
\]
Since \( -\ln |\cos x| = \ln(|\cos x|^{-1}) = \ln(1/|\cos x|) = \ln |\sec x| \), the result of Example 11 can also be written as
\[
\int \tan x \, dx = \ln |\sec x| + C
\]
**General Logarithmic and Exponential Functions**

Formula 6.3.7 expresses a logarithmic function with base \( b \) in terms of the natural logarithmic function:

\[
\log_b x = \frac{\ln x}{\ln b}
\]

Since \( \ln b \) is a constant, we can differentiate as follows:

\[
\frac{d}{dx} (\log_b x) = \frac{\frac{d}{dx} \ln x}{\ln b} = \frac{1}{\ln b} \frac{d}{dx} (\ln x) = \frac{1}{x \ln b}
\]

\[
\frac{d}{dx} (\log_b x) = \frac{1}{x \ln b}
\]

**EXAMPLE 12** Using Formula 6 and the Chain Rule, we get

\[
\frac{d}{dx} \log_{10}(2 + \sin x) = \frac{1}{(2 + \sin x) \ln 10} \frac{d}{dx} (2 + \sin x) = \frac{\cos x}{(2 + \sin x) \ln 10}
\]

From Formula 6 we see one of the main reasons that natural logarithms (logarithms with base \( e \)) are used in calculus: the differentiation formula is simplest when \( b = e \) because \( \ln e = 1 \).

**Exponential Functions with Base \( b \)*** In Section 6.2 we showed that the derivative of the general exponential function \( f(x) = b^x \), \( b > 0 \), is a constant multiple of itself:

\[
f'(x) = f''(0)b^x \quad \text{where} \quad f'(0) = \lim_{h \to 0} \frac{b^h - 1}{h}
\]

We are now in a position to show that the value of the constant is \( f'(0) = \ln b \).

\[
\frac{d}{dx} (b^x) = b^x \ln b
\]

**PROOF** We use the fact that \( e^{\ln b} = b \):

\[
\frac{d}{dx} (b^x) = \frac{d}{dx} (e^{\ln b})^x = \frac{d}{dx} e^{(\ln b)x} = e^{(\ln b)x} \frac{d}{dx} (\ln b)x = (e^{\ln b})^x \ln b = b^x \ln b
\]

In Example 2.7.6 we considered a population of bacteria cells that doubles every hour and we saw that the population after \( t \) hours is \( n = n_0 2^t \), where \( n_0 \) is the initial population. Formula 7 enables us to find the growth rate:

\[
\frac{dn}{dt} = n_0 2^t \ln 2
\]
EXAMPLE 13 Combining Formula 7 with the Chain Rule, we have

\[
\frac{d}{dx} \left(10^{x^2}\right) = 10^{x^2} \ln(10) \frac{d}{dx} \left(x^2\right) = (2 \ln 10)x10^{x^2}
\]

The integration formula that follows from Formula 7 is

\[
\int b^x \, dx = \frac{b^x}{\ln b} + C \quad b \neq 1
\]

EXAMPLE 14

\[
\int_0^5 2^x \, dx = \left[\frac{2^x}{\ln 2}\right]_0^5 = \frac{2^5}{\ln 2} - \frac{2^0}{\ln 2} = \frac{31}{\ln 2}
\]

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called logarithmic differentiation.

EXAMPLE 15 Differentiate \(y = \frac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}\).

SOLUTION We take logarithms of both sides of the equation and use the properties of logarithms to simplify:

\[\ln y = \frac{3}{4} \ln x + \frac{1}{2} \ln(x^2 + 1) - 5 \ln(3x + 2)\]

Differentiating implicitly with respect to \(x\) gives

\[
\frac{1}{y} \frac{dy}{dx} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2 + 1} - 5 \cdot \frac{3}{3x + 2}
\]

Solving for \(dy/dx\), we get

\[
\frac{dy}{dx} = y \left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)
\]

Because we have an explicit expression for \(y\), we can substitute and write

\[
\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5} \left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)
\]

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation \(y = f(x)\) and use the properties of logarithms to simplify.
2. Differentiate implicitly with respect to \(x\).
3. Solve the resulting equation for \(y'\).

If \(f(x) < 0\) for some values of \(x\), then \(\ln f(x)\) is not defined, but we can still use logarithmic differentiation by first writing \(|y| = |f(x)|\) and then using Equation 3. We illustrate this procedure by proving the general version of the Power Rule, as promised in Section 2.3.
The Power Rule  If $n$ is any real number and $f(x) = x^n$, then
\[ f'(x) = nx^{n-1} \]

PROOF  Let $y = x^n$ and use logarithmic differentiation:
\[
\ln |y| = \ln |x|^n = n \ln |x| \quad x \neq 0
\]
Therefore
\[
\frac{y'}{y} = \frac{n}{x}
\]
Hence
\[
y' = n \frac{y}{x} = n \frac{x^n}{x} = nx^{n-1}
\]

You should distinguish carefully between the Power Rule $[(d/dx)x^n = nx^{n-1}]$, where the base is variable and the exponent is constant, and the rule for differentiating exponential functions $[(d/dx)b^x = b^x \ln b]$, where the base is constant and the exponent is variable.

In general there are four cases for exponents and bases:

1. Constant base, constant exponent
   \[ \frac{d}{dx} (b^n) = 0 \quad (b \text{ and } n \text{ are constants}) \]

2. Variable base, constant exponent
   \[ \frac{d}{dx} [f(x)]^n = n[f(x)]^{n-1}f'(x) \]

3. Constant base, variable exponent
   \[ \frac{d}{dx} [b^{g(x)}] = b^{g(x)} (\ln b) g'(x) \]

4. Variable base, variable exponent
   To find $(d/dx) [f(x)]^{g(x)}$, logarithmic differentiation can be used, as in the next example.

EXAMPLE 16  Differentiate $y = x^{\sqrt{x}}$.

SOLUTION 1  Since both the base and the exponent are variable, we use logarithmic differentiation:
\[
\ln y = \ln x^{\sqrt{x}} = \sqrt{x} \ln x
\]
\[
\frac{y'}{y} = \sqrt{x} \cdot \frac{1}{x} + (\ln x) \frac{1}{2\sqrt{x}}
\]
\[
y' = y \left( \frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}} \right) = x^{\sqrt{x}} \left( \frac{2 + \ln x}{2\sqrt{x}} \right)
\]

SOLUTION 2  Another method is to write $x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}}$;
\[
\frac{d}{dx} (x^{\sqrt{x}}) = \frac{d}{dx} (e^{\sqrt{x} \ln x}) = e^{\sqrt{x} \ln x} \frac{d}{dx} (\sqrt{x} \ln x)
\]
\[
= x^{\sqrt{x}} \left( \frac{2 + \ln x}{2\sqrt{x}} \right) \quad \text{(as in Solution 1)}
\]
### The Number e as a Limit

We have shown that if \( f(x) = \ln x \), then \( f'(x) = 1/x \). Thus \( f'(1) = 1 \). We now use this fact to express the number \( e \) as a limit.

From the definition of a derivative as a limit, we have

\[
f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1 + x) - f(1)}{x}
\]

\[
= \lim_{x \to 0} \frac{\ln(1 + x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1 + x)
\]

\[
= \lim_{x \to 0} \ln(1 + x)^{1/x}
\]

Because \( f'(1) = 1 \), we have

\[
\lim_{x \to 0} \ln(1 + x)^{1/x} = 1
\]

Then, by Theorem 1.8.8 and the continuity of the exponential function, we have

\[
e = e^1 = e^{\lim_{x \to 0} \ln(1 + x)^{1/x}} = \lim_{x \to 0} e^{\ln(1 + x)^{1/x}} = \lim_{x \to 0} (1 + x)^{1/x}
\]

Formula 8 is illustrated by the graph of the function \( y = (1 + x)^{1/x} \) in Figure 7 and a table of values for small values of \( x \). This illustrates the fact that, correct to seven decimal places,

\[
e = 2.7182818
\]

If we put \( n = 1/x \) in Formula 8, then \( n \to \infty \) as \( x \to 0^+ \) and so an alternative expression for \( e \) is

\[
e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n
\]

### 6.4 Exercises

1. Explain why the natural logarithmic function \( y = \ln x \) is used much more frequently in calculus than the other logarithmic functions \( y = \log_b x \).

2.–26 Differentiate the function.

2. \( f(x) = x \ln x - x \)

3. \( f(x) = \sin(\ln x) \)

4. \( f(x) = \ln(\sin^2 x) \)

5. \( f(x) = \ln \frac{1}{x} \)

6. \( y = \frac{1}{\ln x} \)

7. \( f(x) = \log_{10}(1 + \cos x) \)

8. \( f(x) = \log_{10} \sqrt{x} \)

9. \( g(x) = \ln(xe^{-2x}) \)

10. \( g(t) = \sqrt{1 + \ln t} \)

11. \( F(t) = (\ln t)^2 \sin t \)

12. \( h(x) = \ln(x + \sqrt{x^2 - 1}) \)

13. \( G(y) = \ln \frac{(2y + 1)^3}{\sqrt{y^2 + 1}} \)

14. \( P(v) = \frac{\ln v}{1 - v} \)

15. \( f(u) = \frac{\ln u}{1 + \ln(2u)} \)

16. \( y = \ln |1 + t - t^3| \)

17. \( f(x) = x^5 + 5^x \)

18. \( g(x) = x \sin(2^x) \)

19. \( T(z) = 2^z \log_2 z \)

20. \( y = \ln(\csc x - \cot x) \)

21. \( y = \ln(e^{-x} + xe^{-x}) \)

22. \( H(z) = \ln \sqrt{\frac{a^2 - z^2}{a^2 + z^2}} \)

23. \( y = \tan[\ln(ax + b)] \)

24. \( y = \log_2(x \log_3 x) \)

25. \( G(x) = 4^{C/x} \)

26. \( F(t) = 3^{\cos 2t} \)
27–30 Find \( y' \) and \( y'' \).
27. \( y = \sqrt{x} \ln x \)
28. \( y = \frac{\ln x}{1 + \ln x} \)
29. \( y = \ln |\sec x| \)
30. \( y = \ln(1 + \ln x) \)

31–34 Differentiate \( f \) and find the domain of \( f \).
31. \( f(x) = \frac{x}{1 - \ln(x - 1)} \)
32. \( f(x) = \sqrt{2 + \ln x} \)
33. \( f(x) = \ln(x^2 - 2x) \)
34. \( f(x) = \ln \ln x \)

35. If \( f(x) = \ln(x + \ln x) \), find \( f'(1) \).
36. If \( f(x) = \cos(\ln x^2) \), find \( f'(1) \).

37–38 Find an equation of the tangent line to the curve at the given point.
37. \( y = \ln(x^2 - 3x + 1) \), (3, 0)
38. \( y = x^2\ln x \), (1, 0)

39. If \( f(x) = \sin x + \ln x \), find \( f'(x) \). Check that your answer is reasonable by comparing the graphs of \( f \) and \( f' \).
40. Find equations of the tangent lines to the curve \( y = (\ln x)/x \) at the points \((1, 0)\) and \((e, 1/e)\). Illustrate by graphing the curve and its tangent lines.

41. Let \( f(x) = cx + \ln(\cos x) \). For what value of \( c \) is \( f''(\pi/4) = 6 \)?
42. Let \( f(x) = \log_b(3x^2 - 2) \). For what value of \( b \) is \( f'(1) = 3 \)?

43–54 Use logarithmic differentiation to find the derivative of the function.
43. \( y = (x^2 + 2)^{(x^4 + 4)^4} \)
44. \( y = \frac{e^{-x} \cos^2 x}{x^2 + x + 1} \)
45. \( y = \sqrt{x^2 - 1} \)
46. \( y = \sqrt{x^2 - 3}(x + 1)^{2/3} \)
47. \( y = x^x \)
48. \( y = x^{\cos x} \)
49. \( y = x^{\sin x} \)
50. \( y = (\sqrt{x})^x \)
51. \( y = (\cos x)^x \)
52. \( y = (\sin x)^{\sin x} \)
53. \( y = (\tan x)^{1/x} \)
54. \( y = (\ln x)^{\cos x} \)

55. Find \( y' \) if \( y = \ln(x^2 + y^2) \).
56. Find \( y' \) if \( x^y = y^x \).
57. Find a formula for \( f^{(n)}(x) \) if \( f(x) = \ln(x + 1) \).

58. Find \( \frac{d^9}{dx^9}(x^4 \ln x) \).

59–60 Use a graph to estimate the roots of the equation correct to one decimal place. Then use these estimates as the initial approximations in Newton’s method to find the roots correct to six decimal places.
59. \( (x - 4)^2 = \ln x \)
60. \( \ln(4 - x^2) = x \)

61. Find the intervals of concavity and the inflection points of the function \( f(x) = (\ln x)/\sqrt{x} \).
62. Find the absolute minimum value of the function \( f(x) = x \ln x \).

63–66 Discuss the curve under the guidelines of Section 3.5.
63. \( y = \ln(\sin x) \)
64. \( y = \ln(\tan^2 x) \)
65. \( y = \ln(1 + x^2) \)
66. \( y = \ln(1 + x^3) \)

67. If \( f(x) = \ln(2x + x \sin x) \), use the graphs of \( f, f' \), and \( f'' \) to estimate the intervals of increase and the inflection points of \( f \) on the interval \((0, 15)\).
68. Investigate the family of curves \( f(x) = \ln(x^2 + c) \). What happens to the inflection points and asymptotes as \( c \) changes? Graph several members of the family to illustrate what you discover.
69. The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The following data describe the charge \( Q \) remaining on the capacitor (measured in microcoulombs, \( \mu C \)) at time \( t \) (measured in seconds).

<table>
<thead>
<tr>
<th>( t )</th>
<th>0.00</th>
<th>0.02</th>
<th>0.04</th>
<th>0.06</th>
<th>0.08</th>
<th>0.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q )</td>
<td>100.00</td>
<td>81.87</td>
<td>67.03</td>
<td>54.88</td>
<td>44.93</td>
<td>36.76</td>
</tr>
</tbody>
</table>

(a) Use a graphing calculator or computer to find an exponential model for the charge.
(b) The derivative \( Q'(t) \) represents the electric current (measured in microamperes, \( \mu A \)) flowing from the capacitor to the flash bulb. Use part (a) to estimate the current when \( t = 0.04 \) s. Compare with the result of Example 1.4.2.

70. The table gives the US population from 1790 to 1860.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
<th>Year</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1790</td>
<td>3,929,000</td>
<td>1830</td>
<td>12,861,000</td>
</tr>
<tr>
<td>1800</td>
<td>5,308,000</td>
<td>1840</td>
<td>17,063,000</td>
</tr>
<tr>
<td>1810</td>
<td>7,240,000</td>
<td>1850</td>
<td>23,192,000</td>
</tr>
<tr>
<td>1820</td>
<td>9,639,000</td>
<td>1860</td>
<td>31,443,000</td>
</tr>
</tbody>
</table>

(a) Use a graphing calculator or computer to fit an exponential function to the data. Graph the data points and the exponential model. How good is the fit?
(b) Estimate the rates of population growth in 1800 and 1850 by averaging slopes of secant lines.
(c) Use the exponential model in part (a) to estimate the rates of growth in 1800 and 1850. Compare these estimates with the ones in part (b).
(d) Use the exponential model to predict the population in 1870. Compare with the actual population of 38,558,000. Can you explain the discrepancy?

71–82 Evaluate the integral.

71. \( \int_2^4 \frac{3}{x} \, dx \)

72. \( \int_0^3 \frac{dx}{5x + 1} \)

73. \( \int_1^2 \frac{dt}{8 - 3t} \)

74. \( \int_0^\pi \left( \sqrt{x} + \frac{1}{\sqrt{x}} \right)^2 \, dx \)

75. \( \int_1^e x^2 + x + 1 \, dx \)

76. \( \int_1^e \frac{\cos(\ln t)}{t} \, dt \)

77. \( \int \frac{(\ln x)^2}{x} \, dx \)

78. \( \int \frac{\cos x}{2 + \sin x} \, dx \)

79. \( \int \frac{\sin 2x}{1 + \cos^2 x} \, dx \)

80. \( \int e^{-x} \, dx \)

81. \( \int_0^4 2^x \, dx \)

82. \( \int x^2 \ln x \, dx \)

83. Show that \( \int \cot x \, dx = \ln |\sin x| + C \) by (a) differentiating the right side of the equation and (b) using the method of Example 11.

84. Sketch the region enclosed by the curves

\[ y = \frac{\ln x}{x} \quad \text{and} \quad y = \frac{(\ln x)^2}{x} \]

and find its area.

85. Find the volume of the solid obtained by rotating the region under the curve

\[ y = \frac{1}{\sqrt{x} + 1} \]

from 0 to 1 about the \( x \)-axis.

86. Find the volume of the solid obtained by rotating the region under the curve

\[ y = \frac{1}{x^2 + 1} \]

from 0 to 3 about the \( y \)-axis.

87. The work done by a gas when it expands from volume \( V_1 \) to volume \( V_2 \) is \( W = \int_{V_1}^{V_2} P \, dV \), where \( P = P(V) \) is the pressure as a function of the volume \( V \). (See Exercise 5.4.29.) Boyle’s Law states that when a quantity of gas expands at constant temperature, \( PV = C \), where \( C \) is a constant. If the initial volume is 600 cm\(^3\) and the initial pressure is 150 kPa, find the work done by the gas when it expands at constant temperature to 1000 cm\(^3\).

88. Find \( f \) if \( f''(x) = x^{-2}, \ x > 0, \ f(1) = 0, \ \text{and} \ f(2) = 0. \)

89. If \( g \) is the inverse function of \( f(x) = 2x + \ln x \), find \( g'(2). \)

90. If \( f(x) = e^x + \ln x \) and \( h(x) = f^{-1}(x) \), find \( h(e). \)

91. For what values of \( m \) do the line \( y = mx \) and the curve \( y = x/(x^2 + 1) \) enclose a region? Find the area of the region.

92. (a) Find the linear approximation to \( f(x) = \ln x \) near 1.

(b) Illustrate part (a) by graphing \( f \) and its linearization.

(c) For what values of \( x \) is the linear approximation accurate to within 0.1?

93. Use the definition of derivative to prove that

\[ \lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1 \]

94. Show that \( \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = e^x \) for any \( x > 0. \)

### 6.2* The Natural Logarithmic Function

If your instructor has assigned Sections 6.2–6.4 (pp. 408–438), you need not read Sections 6.2*, 6.3*, and 6.4* (pp. 438–465).

In this section we define the natural logarithm as an integral and then show that it obeys the usual laws of logarithms. The Fundamental Theorem makes it easy to differentiate this function.

**Definition** The natural logarithmic function is the function defined by

\[ \ln x = \int_1^x \frac{1}{t} \, dt \quad x > 0 \]